Transient, global-in-time, convergent iterative coupling of acoustic BEM and elastic FEM

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Abstract. Motivated by ongoing studies in naval engineering, we address a domain decomposition approach for the numerical solution of transient problems involving submerged elastic structures subjected to pressure waves. The fluid region is treated with boundary elements (BEM), formulated in discrete time by the Convolution Quadrature method (CQM), a classical finite element method (FEM) is used for the solid, and the two media are coupled globally in time via iteratively solving Robin problems in each domain. We prove the convergence of this iterative algorithm, and demonstrate it on 2D numerical examples.

Keywords: Acoustic-elastic coupling, iterative methods, convolution quadrature

Introduction Motivated by an ongoing applied study in naval engineering, this work addresses a domain decomposition approach for the numerical solution of large-scale transient problems involving submerged structures subjected to pressure shock waves caused by remote underwater explosions and travelling in the surrounding fluid at short time scales.

Treating such situations as transient acoustic scattering by elastic obstacles is valid from an industry standpoint. The fluid response is formulated in the discrete-time domain, using a Z-BEM approach that combines a fast boundary element method (BEM) implemented in the in-house fast multipole solver COFFEE in the Laplace domain [2] with the convolution quadrature method (CQM), see details in [6]. Efficiency requires the Z-BEM formulation to be applied for the entire time interval at once. The structure response is modelled using standard finite element and time-stepping methods.

Earlier attempts at iteratively solving the resulting BEM-FEM coupling by alternating Neumann solutions in each domain have failed due to non-convergence. We hence focus in this work on an iterative algorithm based on Robin boundary conditions for the coupled elastodynamic-acoustic problem.

Acoustic-elastic scattering problem The fluid-structure interaction (FSI) problem is:

$$\begin{aligned} -\Delta\phi(t,\mathbf{x}) + \frac{1}{c^2}\partial_{tt}\phi(t,\mathbf{x}) = 0 &\quad \text{in } \Omega_{\rm f}\times[0,T], \\ -div\sigma[\mathbf{u}] + \rho_{\rm s}\partial_{tt}\mathbf{u} = \mathbf{0} &\quad \text{in } \Omega_{\rm s}\times[0,T], \\ \mathbf{t} = \rho_{\rm f}\partial_t\phi\mathbf{n} + \rho_{\rm f}\partial_t\phi^{inc}\mathbf{n} &\quad \text{in } \Gamma\times[0,T], \\ v = \partial_t\mathbf{u}.\mathbf{n} - \partial_n\phi^{inc} &\quad \text{in } \Gamma\times[0,T] \end{aligned} \tag{1}$$

and initial-rest conditions, where $\Omega_{\rm s}$ is an elastic solid surrounded by the acoustic fluid domain $\Omega_{\rm f} := \mathbb{R}^3 \setminus \overline{\Omega_{\rm s}}$ and $\Gamma = \partial\Omega_{\rm s}$. The fluid velocity potential is decomposed into a given incident field ϕ^{inc} and an unknown scattered field ϕ (with velocity $\mathbf{v} = \nabla\phi$ and pressure $p = -\rho_{\rm f}\partial_t\phi$). The total solid displacement is $\mathbf{u}$, and $\mathbf{t} := \sigma[\mathbf{u}].\mathbf{n}$ is the stress vector. Problem (1) is well-posed [5], and so is its frequency-domain counterpart [1].

Robin-Robin iterations Inspired by the convergent iterative domain decomposition (DD) approach of [3] for frequency-domain Helmholtz problems, based on Robin solutions in each domain, we propose for solving the problem (1) an iterative DD method based on a sequence of Robin initial-boundary value problems, iteration $i+1$ being defined by boundary conditions of the form

$$\begin{aligned} -\rho_{\rm f}\partial_t\phi^{i+1} + k_{\rm f}\partial_n\phi^{i+1} = g^{i+1} &\quad \text{in } \Gamma\times[0,T] \\ \mathbf{t}^{i+1} + k_{\rm s}\partial_t\mathbf{u}^{i+1} = \mathbf{g}^{i+1} &\quad \text{in } \Gamma\times[0,T] \end{aligned} \tag{2a}$$

where $k_{\rm f}, k_{\rm s} > 0$ are adjustable parameters and the Robin data g^{i+1} (for the fluid) and $\mathbf{g}^{i+1}$ (for the solid) are defined in terms of the solution traces from iteration $i-1$ by

$$\begin{aligned} g^{i+1} &= \mathbf{n}\cdot(-\mathbf{t}^i + k\partial_t\mathbf{u}^i) + \rho_{\rm f}\partial_t\phi^{\rm inc} + k\partial_n\phi^{\rm inc} \\ \mathbf{g}^{i+1} &= (\mathbf{n}\otimes\mathbf{n} - \mathbf{I})\cdot(\mathbf{t}^i - k\partial_t\mathbf{u}^i) \\ &\quad +\mathbf{n}(\rho_{\rm f}\partial_t + k\partial_n)\phi^i + \mathbf{n}(\rho_{\rm f}\partial_t + k\partial_n)\phi^{\rm inc} \end{aligned} \tag{2b}$$

for the simplest case $k_{\rm f} = k_{\rm s}$ (corresponding formulas for $k_{\rm f} \neq k_{\rm s}$ being also available). The RR iterations (2a,b) are started by setting $\phi^0 =$

$\mathbf{u}^0 = 0$. With the terminology used in [3], the incoming traces of iterate $i+1$ are related by (2a,b) to the outgoing traces of iterate i.

Convergence We prove that the RR iterations defined by (2a,b) converge; more precisely, we show that:

(a) If $\partial_t \phi^{\rm inc}, \partial_n \phi^{\rm inc} \in L^2(\Gamma\times[0,T])$, the incoming and outgoing traces of each iterate generated by (2a,b) also are in $L^2(\Gamma\times[0,T])$, by a clasical energy estimate argument [4].

(b) the $C^0([0,T], H^1(\Omega_{\rm f}) \cap C^1([0,T], L^2(\Omega_{\rm f})$ norm of the error field $\phi^i - \phi$ vanishes in the limit $i\to\infty$, and similarly for the error field $\mathbf{u}^i - \mathbf{u}$. This is proved by showing that the norms of all error iterates have a sum bounded by a positive constant.

In addition, transient BEM-FEM coupling based instead on Neumann-Neumann iterations would be problematic as energy estimates indicate that each iteration degrades the regularity of boundary traces (unlike in the elliptic case).

We are also currently working on obtaining data-to-solution solvability mappings for problem (1) that supplement the result of [5] and show that the FSI solution has all (co)normal derivatives and velocity traces in $L^2(\Gamma\times[0,T])$ if $\partial_t \phi^{\rm inc}, \partial_n \phi^{\rm inc} \in H^1([0,T], L^2(\Gamma))$, in which case the FSI solution is attainable by RR iterates.

Numerical results Numerical examples for the FSI problem (1) for 2D geometries solved using FEM-BEM iterative coupling will be presented, demonstrating convergence as well as enhancements such as a relaxed version of (2a,b) and Aitken-type convergence acceleration.

Consider for example the case of an elastic ring subjected to an internal radially-symmetric transient pressure and immersed in an acoustic fluid. This problem, which can be put in the format (1) upon splitting the solid (rather than the fluid) response, is useful for validation purposes as it has a radially-symmetric analytical solution against which to compare the response computed using the 2D BEM-FEM iterative coupling method (Figure 1). Figure 2 shows the convergence of the relative solution error (in space-time L^2 norm) as iterations progress.

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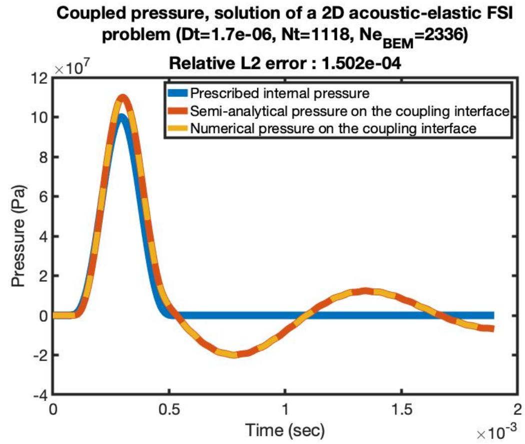

Figure 1: Pressure radiated by an elastic annulus: comparison of solution obtained by iterative FEM-BEM coupling with an analytical solution.

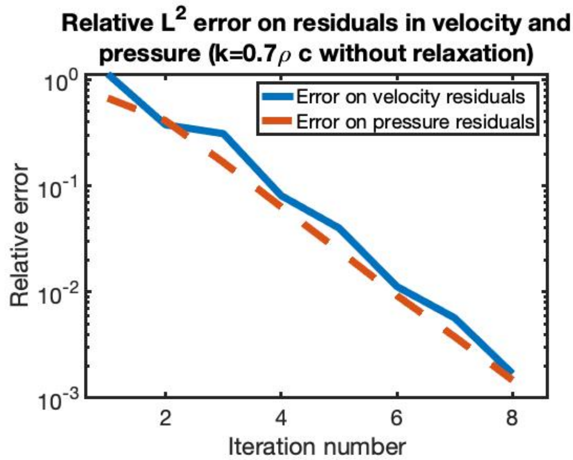

Figure 2: Convergence of the relative transmission error with the iteration number i: $\|\partial_t u_i - v_i\|^2_{L_2} \,/\, \|v_i\|^2_{L_2}$.

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