Dirichlet-to-Neumann Coupling for 2D-1D Time-Dependent Wave Problems

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Abstract

The coupling of two-dimensional (2D) and one-dimensional (1D) models to form a single hybrid 2D-1D model is considered, for the time-dependent linear scalar wave equation. The 1D model is an approximation of a part of the original 2D model where the solution behaves approximately in a 1D way. This hybrid model, if designed properly, is a more efficient way to solve the problem compared to the full 2D model. The 2D-1D coupling is done using the Dirichlet-to-Neumann (DtN) map associated with the 1D part of the problem. We shall discuss two ways in which the DtN coupling can be done: in one of them the 2D and 1D problems exchange information in each time step, whereas in the other the two problems are solved independently. The well-posedness of the hybrid problem as well as the coupling error are discussed, and numerical examples are presented.

Keywords: coupling, mixed-dimensional, 2D-1D, Dirichlet-to-Neumann, DtN

1 Introduction

A reoccurring theme in computational mechanics in recent years is the need to reduce the size of large discrete models. One type of such a reduction is spatial dimensional reduction, which one may perform in cases where the solution in some region of a high-dimensional (highD) computational domain, say two-dimensional (2D), behaves in a low-dimensional (lowD) way, say one-dimensional (1D). There are several scenarios where this could be the case. Most significant is the scenario where the solution in a certain region behaves in a way that is weakly (or hardly) dependent on a certain coordinate, relative to the other coordinates. Another possible scenario is when we are interested in the solution within a geometrically slender region. In this case we might be interested in the lateral average of the solution within this region rather than in its lateral distribution. Alternatively we might already know the nature of the lateral distribution of the solution within this region and wish to know the axial distribution. In these cases, the lateral dimension is the dimension we would eliminate, resulting in a mixed-dimensional model. Fig. 1 describes the typical characteristics of the highD and lowD sub-models.

The motivation in constructing a mixed-dimensional model comes from the fact that solving the problem in its highD form everywhere may require a very large computational effort. The idea is thus to partly reduce the spatial dimension of the problem in order to obtain a hybrid model which is much more efficient. Fields of application where mixed-dimensional coupling is of special interest include, among others, the following:

- *Blood-flow analysis.* Typically the HighD model corresponds to a specific blood vessel of interest in the human body and the LowD model corresponds to the rest of the blood system. An example can be found in [1].
- *Hydrological and geophysical flow models.* Here the LowD region represents a collection of channel-like entities (rivers, flood streams, etc.) and the HighD region is that of a large water body (a river delta, a lake, etc.).
- *Elastic structures.* Typically the LowD model consists of the slender parts of the structure that have rod- or beam- or plate- or shell-like behavior, and which constitute most of the structure volume, while the HighD parts are the regions that have to be modeled as 3D elastic bodies. Panasenko *et al.* have developed an asymptotic-variational approach for such structural problems, under static conditions. See, e.g., [2].

Our focus in this work is the latter application, although the coupling methods developed are general and may be useful for other applications as well. In this talk, we apply 2D-1D cou-

highD	lowD
Region of interest	'Outer region' – less of interest in itself, but affects the behavior in the highD region significantly
General geometry	Typically a slender region
highD behavior of the solution	lowD behavior of the solution (e.g., the solution varies only slightly in the lateral direction, or only the average in that direction is of importance)
PDE, BCs, ICs: faithful to the highD physics	PDE, BCs, ICs are lowD approximations of their highD counterparts (reduced model)

Figure 1: The characteristics of the highD and lowD sub-models

pling to wave problems governed by the scalar wave equation. The method of coupling chosen here is one which makes use of the Dirichlet-to-Neumann (DtN) map associated with the 1D part of the problem.

2 The hybrid model

After splitting the given problem into a 2D part and a 1D part, thus creating a hybrid 2D-1D model, we need to consider the interface conditions imposed at the continuous level (i.e., before any discretization takes place). There are two interface conditions: on the wave function u (the acoustic pressure) and on its "flux" (the normal derivative $\partial u/\partial n$). We discuss the well-posedness of the hybrid problem using these two continuity conditions.

3 DtN mixed-dimensional coupling

As mentioned above, the 2D-1D coupling is done using the Dirichlet-to-Neumann (DtN) map associated with the 1D part of the problem. The DtN map relates the primary variable to the "flux" on an interface. In the mixed-dimensional context, the DtN method couples the highD and lowD models by enforcing the continuity of the DtN map across the highD-lowD interface. The rationale is that rather than insisting that both the primary variable and the flux be continuous across the interface, it makes sense to have the mapping between them continuous. The DtN method was used in [3] for the Helmholtz equation, for the case where the 2D behavior is persistent in the 1D region. In the time-harmonic case (the frequency domain), the DtN method has been found to be especially effective [3, 4]. Here we use the DtN map coupling in the time domain, which is a major extension.

We shall discuss two ways in which the DtN coupling can be done: in one of them the 2D and 1D problems exchange information in each time step [5], whereas in the other the two problems are solved independently.

By experimenting with various numerical examples, we shall investigate the performance of these methods and estimate the errors that each of them generates.

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