A space–time continuous and coercive variational formulation for the wave equation

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Abstract

In recent years, several space–time formulations and discretizations for the wave equation have been proposed. With "space–time" we mean that they are posed in the space–time cylinder, as opposed to semi-discretizations in space combined with time stepping. To our knowledge, no continuous and coercive (i.e. sign-definite) variational formulation has been proposed yet. The aim of this paper is to fill this gap proposing a Lax–Milgram formulation on the space–time cylinder for the wave equation with impedance boundary conditions. In order to do this, we follow a strategy previously adopted for Helmholtz problems, which relies on Morawetz identities and multipliers. From this, a continuous and coercive bilinear form and a linear form on an appropriate space can be constructed. We also show the explicit coercivity constant.

Keywords: wave equation, variational problem, coercive, sign-definite, space–time

1 Introduction and notation

We consider an initial–boundary value problem (IBVP) for the wave equation with impedance boundary conditions:

$$\begin{cases} u_{tt} - c^2\Delta u = f & \text{on } Q = (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} + \frac{1}{\theta c} u_t = g & \text{on } \Sigma = (0,T)\times\partial\Omega, \\ u = u_0 & \text{on } \Omega_0 = \{0\}\times\Omega, \\ u_t = u_1 & \text{on } \Omega_0. \end{cases} \tag{1}$$

Here $_t$ denotes time differentiation, $\Omega \subset \mathbb{R}^d$ is an open, bounded, Lipschitz domain, $c > 0$ and $\theta > 0$ are constants, and f, g, u_0 and u_1 are appropriate functions defined on Q, Σ and Ω_0, respectively. Let also $\Omega_T = \{T\}\times\Omega$.

Assume that exist $L > 0$ and $\delta > 0$ such that $|\mathbf{x}| \le L$ for all $\mathbf{x} \in \Omega$ and $\mathbf{x}\cdot\mathbf{n}(\mathbf{x}) \ge \delta L$ for all $\mathbf{x} \in \partial\Omega$, where $\mathbf{n}(\mathbf{x})$ is the normal to $\partial\Omega$ at $\mathbf{x}$. This implies that Ω is star-shaped with respect to the ball $B_{\delta L}(\mathbf{0})$.

We define the seminorm

$$\begin{aligned} \|v\|_V^2 :=& \|v_t\|_Q^2 + c^2\|\nabla v\|_Q^2 + T^2\|u_{tt} - c^2\Delta u\|_Q^2 \\ &+ L\|v_t\|_\Sigma^2 + c^2 L\|\nabla v\|_\Sigma^2 \\ &+ c^2 T\|\nabla v\|_{\Omega_0}^2 + T\|v_t\|_{\Omega_0}^2 \\ &+ c^2 T\|\nabla v\|_{\Omega_T}^2 + T\|v_t\|_{\Omega_T}^2. \end{aligned}$$

This is a norm on $C^\infty_*(\overline{Q}) := \{u \in C^\infty(\overline{Q}) : \int_{\Omega_0} u = 0\}$ and on $V := \overline{C^\infty_*(\overline{Q})}^{\|\cdot\|_V}$. We assume that the initial datum u_0 has zero average in Ω_0. Problems with more general data can be treated by simply adding a constant to the solution.

2 Abstract framework

We introduce an abstract setting that includes coercive formulations for both the wave and the Helmholtz equations, generalising the approach in [2]. Consider a linear boundary value problem (BVP):

$$\begin{cases} \mathcal{L}u = f & \text{on } D \subset \mathbb{R}^n, \\ \mathcal{B}u = g & \text{on } \partial D, \end{cases} \tag{2}$$

where the operators $\mathcal{L}$ and $\mathcal{B}$ act on the (either real or complex) Hilbert space $H \subset L^2(Q)$ with norm $\|\cdot\|_H$. We want to write the BVP (2) as a variational problem

$$\text{find } u \in H\text{: } b(u,v) = F(v) \qquad \forall v \in H, \tag{3}$$

whose sesquilinear form b is continuous and coercive (sign-definite) in H, so that the problem is well-posed by Lax–Milgram theorem. To achieve this, we need an operator $\mathcal{M} : H \to L^2(D)$ (the "Morawetz multiplier"), and two sesquilinear forms X and G that are continuous on H and such that the following decomposition holds (the "integrated Morawetz identity"):

$$\int_D (\mathcal{L}u\overline{\mathcal{M}v} + \mathcal{M}u\overline{\mathcal{L}v}) = X(u,v) + G(u,v). \tag{4}$$

We require that $G(u,v) = G_g(v)$ for $G_g \in H^*$ when $\mathcal{B}u = g$, so that $G(\cdot,\cdot)$ collects the terms coming from the boundary conditions.

For $A > 0$, we define

$$b(u,v) := -X(u,v) + \int_D \left[\mathcal{M}u\overline{\mathcal{L}v} + A\mathcal{L}u\overline{\mathcal{L}v}\right],$$

$$F(v) := G_g(v) + \int_D \left[-f\overline{\mathcal{M}v} + Af\overline{\mathcal{L}v}\right].$$

It is immediate to check that, with these definitions, a solution of the BVP (2) solves (3).

The coercivity of b is equivalent to the existence of $C > 0$ such that

$$\frac{1}{2}\big[G(u,u) - X(u,u)\big] + A\|\mathcal{L}u\|^2_{L^2(D)} \geq C\|u\|^2_H \tag{5}$$

for all $u \in H$. Indeed, from (4) it holds that

$$\Re \int_D \mathcal{L}u\overline{\mathcal{M}u} = \frac{1}{2}\big[X(u,u) + G(u,u)\big],$$

therefore $\Re b(u,u)$ is equal to

$$\int_D A|\mathcal{L}u|^2 + \frac{1}{2}\left[X(u,u) + G(u,u)\right] - X(u,u),$$

and $\Re b(u,u) \geq C\|u\|^2_H$ is (5).

By setting $\mathcal{L}u = \Delta u + k^2 u$, $\mathcal{B}u = \frac{\partial u}{\partial n} - ik\theta u$, $\mathcal{M}u = \mathbf{x}\cdot\nabla u + \alpha u - ik\beta u$ for suitable parameters $\alpha, \beta \in \mathbb{R}$, one recovers the coercive formulation for Helmholtz impedance BVPs described in [2].

3 A space–time formulation for the wave equation

Existing space–time formulations for second-order hyperbolic IBVPs, e.g. [3], do not satisfy Lax–Milgram assumptions. We formulate the IBVP (1) in the abstract framework described above. We let $H = V$, $D = Q$, $\mathcal{L}u := u_{tt} - c^2\Delta u$, $\mathcal{B}$ be the operator collecting impedance conditions on Σ and initial conditions on Ω_0, and choose the Morawetz multiplier as

$$(\mathcal{M}u)(t,\mathbf{x}) := -\xi\mathbf{x}\cdot\nabla u + \beta(t - T^*)u_t,$$

for all $(t,\mathbf{x}) \in Q$, where $\beta, \xi > 0$ and $T^* = \nu T$, with $\nu > 1$ are fixed parameters.

We define X and G in (4) as

$$\begin{aligned} X(u,v) &:= \int_{\Omega_T} \beta(T - T^*)(u_t v_t + c^2\nabla u\cdot\nabla v) \\ &- \int_{\Omega_T} \xi\mathbf{x}\cdot(u_t\nabla v + \nabla u v_t) \\ &- \int_Q \big(u_t v_t(\beta + \xi d) + c^2\nabla u\cdot\nabla v(\beta + 2\xi - \xi d)\big) \\ &- \int_\Sigma c^2\left[\mathcal{M}u\frac{\partial v}{\partial n} - \frac{1}{\theta c}u_t\mathcal{M}v\right] \\ &- \int_\Sigma \xi\mathbf{x}\cdot\mathbf{n}\left[c^2\nabla u\cdot\nabla v - u_t v_t\right], \end{aligned}$$

$$\begin{aligned} G(u,v) &:= \int_{\Omega_0} \beta T^*(u_t v_t + c^2\nabla u\cdot\nabla v) \\ &+ \int_{\Omega_0} \xi\mathbf{x}\cdot(u_t\nabla v + \nabla u v_t) - \int_\Sigma c^2\left[\frac{1}{\theta c}u_t + \frac{\partial u}{\partial n}\right]\mathcal{M}v. \end{aligned}$$

Then one can compute $G_g(\cdot)$ (by substituting u_0, u_1 and g in $G(\cdot,\cdot)$), $b(\cdot,\cdot)$ and $F(\cdot)$ as defined above. Their continuity with respect to the seminorm $\|\cdot\|_V$ follows from repeated use of the Cauchy–Schwarz inequality, though it is not explicity computed here.

4 Coercivity result

As observed in Section 2, to prove coercivity it is sufficient to prove (5). Indeed this is true, and the coercivity constant is reported explicitly. Only simple vector-calculus identities are used in the proof. Recall that $0 < \delta < 1$ measures the "star-shapedness" of Ω.

Lemma. *If*

$$\beta \geq \begin{cases} \xi(d-1) \\ \xi\frac{1}{\nu-1}\left(\frac{L}{cT} + 1\right) \\ \xi\frac{1}{\nu-1}\frac{L}{cT}\left(\delta\theta + \frac{1}{\delta\theta}\right) \end{cases}$$

and $\xi > 0$ and for any $\nu > 1$, then:

$$b(v,v) \geq \min\left\{\frac{\xi\delta}{4}, \frac{A}{T^2}\right\}\|v\|^2_V \qquad \forall v \in V.$$

Therefore the corresponding problem (3) is well-posed and any conforming Galerkin discretization is well-posed and quasi-optimal.

Standard $H^1(Q)$-conforming discretizations such as piecewise-(bi)linear finite elements are not acceptable for this formulation. Indeed they are not conforming in V, because of the $\|u_{tt} - c^2\Delta u\|_Q$ term in the norm, while $C^1(\overline{Q})$-conforming schemes can be used.

The complete proof of the lemma and more details will be available in [1].

References

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