High frequency analysis of the Dirichlet to Neumann operator for the Helmholtz equation on a coated cylinder or elliptic cylinder

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Abstract

This paper studies the Dirichlet to Neumann operator for problems in $\Omega_0^c \subset \mathbb{R}^3$, $\Omega = \Omega_0 \cup L$, where Ω_0 is a perfectly conducting cylinder $S(0,R)\times\mathbb{R}$ or an elliptic cylinder $E\times\mathbb{R}$ coated with a layer L. Obtaining the exact expression of this operator leads to the high frequency behavior of this operator, in the elliptic and in the hyperbolic regions of the dual space (that we are able to characterize). The global high frequency behavior (extension of the classical microlocal expression of this operator) is given, the second term of the high frequency expansion is derived in a rigorous way.

Keywords: Helmholtz equation, Dirichlet to Neumann operator, high frequency expansion.

1 Introduction

This paper deals with a Helmholtz representation of solutions of the Maxwell equation in a dielectric material L. The dielectric constants ϵ, μ satisfy $\epsilon\mu \notin \mathbb{R}_+$. As the problem for $u \in H^1(L)$, $u_0 \in H^{\frac{1}{2}}(\partial\Omega)$:

$$\begin{cases} (\Delta + \epsilon\mu\omega^2)u = 0, (x,y,z) \in L \\ \gamma_{\Omega_0} u = 0 \\ \gamma_\Omega u = u_0 \end{cases}$$

has a unique solution U [2], the well defined operator $u_0 \to \partial_n U$ from $H^{\frac{1}{2}}(L)$ to $H^{-\frac{1}{2}}(L)$ is called the Dirichlet to Neumann operator (DTN).

In the classical case where L is a layer $\mathbb{R}^2 \times (-l_0)$, the solution of this problem is explicit and gives rise to the classical Dirichlet to Neumann operator for a plane layer as a Fourier multiplier on $\mathcal{S}(\mathbb{R}^2)$ (including $k_1^2 + k_2^2 > \Re\epsilon\mu\omega^2$):

$$DTN(\omega, \vec{k}_{tan}) = \frac{\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2}}{\tan(\sqrt{\epsilon\mu\omega^2 - k_1^2 - k_2^2 l})}.$$

Its generalization to a curved layer [5] was derived with a local approximation of $\partial\Omega$, which gives rise to a representation of the Laplace operator as

$$\frac{\partial^2}{\partial n^2} + g_{11}(x_1,x_2,n)\partial^2_{x_1^2} + 2g_{12}(x_1,x_2,n)\partial^2_{x_1x_2} + g_{22}(x_1,x_2,n)\partial^2_{x_2^2} + l.o.t = \Delta$$

in a semi-geodesic system of coordinates ($\partial\Omega \cap V(x^0) = \{n=0\} \cap V(x^0) = \{M(x_1,x_2), (x_1,x_2) \in W(0,0)\}$), and the DTN operator is equivalent locally to a pseudodifferential operator [4], of principal symbol (ωl constant)

$$\omega \frac{\nu(x,\eta)}{\tan \nu(x,\eta)\omega l)},$$

where $\nu(x,\eta)$, $\Im\nu > 0$ is equal to $\sqrt{\epsilon\mu - (g_{11}\eta_1^2 + 2g_{12}\eta_1\eta_2 + g_{22}\eta_2^2)(x_1,x_2,0)}$.

Expressions of the other terms of this symbol are extremely tedious to obtain. Moreover this expression only gives a local analysis of the Dirichlet to Neumann operator which is a global operator on the boundary $\partial\Omega$. The aim of this Note is to give the global expression of the DTN in two particular cases and we derive the two first terms of its high frequency expansion.

2 Exact expression of the DTN in two cases

It is classical [3], [6] to obtain the DTN as a Fourier multiplier when $\Omega_0 = S(0,r_0)\times\mathbb{R}$ and $\Omega = S(0,R)\times\mathbb{R}$ thanks to the representation of the solutions of

$$(\Delta + \epsilon\mu\omega^2 - k^2)u = 0, \{r_0 \le r \le R, \theta \in [0,2\pi)\}$$

in polar coordinates. Assuming

$$\hat{u}_0(\theta,k) = \sum_{n\in\mathbb{Z}} u(n,k)e^{in\theta},$$

one obtains, using $k_3 = \sqrt{\epsilon\mu\omega^2 - k^2}$

$$\mathcal{F}(DTN(u_0))(\theta,k) = \sum_{n\in\mathbb{Z}} D(k,n)u(n,k)e^{in\theta}$$

with $D(k,n) = \frac{J_n'(k_3R)Y_n(r_0) - Y_n'(k_3R)J_n(k_3r_0)}{J_n(k_3R)Y_n(k_3r_0) - Y_n(k_3R)J_n(k_3r_0)}$. Note that one can replace any pair of independent solutions of the Bessel equation by any other pair.

The case $\Omega_0 = \{(x,y,z), \frac{x^2}{\cosh^2 u_0} + \frac{y^2}{\sinh^2 u_0} \leq \rho^2\}$ and $\Omega = \{(x,y,z), \frac{x^2}{\cosh^2 u_1} + \frac{y^2}{\sinh^2 u_1} \leq \rho^2\}$ (a layer between two ellipses with the same focal points) is less known. It is known from Floquet theory that the periodic Mathieu operator $-\frac{d^2}{dv^2} + \cos 2v$ has Bloch eigenvalues $a(n,k), b(n,k)$ associated with the normalized modes $cn(v), sn(v)$ (generalization of the $\cos n\theta, \sin n\theta$ modes for the operator $\frac{d^2}{dv^2}$). The set $\{cn, sn\}$ is a basis of $L^2([0,2\pi])$.
Denote by $F^a_{n,k}, G^a_{n,k}$ a pair of independent solutions of the modified Mathieu equation $\frac{d^2}{du^2}\phi + (a(n,k) - \frac{1}{2}\rho^2 k_3^2 \cosh 2u)\phi = 0$ (and a similar definition replacing $a(n,k)$ by $b(n,k)$).
For $\hat{u}_0(n,k,v) = u(n,k)cn(v)$, the solution U of the Helmholtz equation satisfying the boundary conditions
$U(u_1,v) = cn(v)$ at $u = u_1$ and $U(u_0,v) = cn(v)$ at $u = u_0$ is
$\frac{F^a(n,k)(u)G^a(n,k)(u_0) - G^a(n,k)(u)F^a(n,k)(u_0)}{F^a(n,k)(u_1)G^a(n,k)(u_0) - G^a(n,k)(u_1)F^a(n,k)(u_0)} cn(v)$.
It belongs to $H^1(\Omega - \Omega_0)$ even if the function $F^a(n,k)(u)cn(v)$ is not in $\mathcal{S}'$. Note that, as $\vec{n} = \frac{(\sinh u_1 \cos v, \cosh u_1 \sin v)}{\sqrt{\sin^2 v + \sinh^2 u_1}}$, $\partial_n f = \frac{1}{\rho\sqrt{\sin^2 v + \sinh^2 u_1}} \partial_u f$.

3 High frequency estimate of the DTN operator

Using precise asymptotic expansions of the Bessel and modified Mathieu functions [1], we prove the two theorems:

Theorem 1 *Consider $\omega \to +\infty$ and assume that $\frac{n}{\omega} \in [c_1, c_2], c_1 > 0$ (**high frequency regime for the Fourier index**). Denote by $\eta = \frac{k}{\omega}$.*

1. *In the case $k^2 + \frac{n^2}{R^2} < \Re\epsilon\mu\omega^2$ (called the hyperbolic case), the symbol of the Dirichlet to Neumann operator is, up to $O(\omega^{-1})$*

$$i\omega\sqrt{\epsilon\mu - \eta^2 - \frac{n^2}{\omega^2 R^2}} - \frac{1}{2R}\frac{\epsilon\mu - \eta^2}{\epsilon\mu - \eta^2 - \frac{n^2}{\omega^2 R^2}}.$$

2. *In the case $k^2 + \frac{n^2}{R^2} > \Re\epsilon\mu\omega^2$ (called the elliptic case), the symbol of the Dirichlet to Neumann operator is, up to $O(\omega^{-1})$*

$$\omega\sqrt{\epsilon\mu - \eta^2 - \frac{n^2}{\omega^2 R^2}} - \frac{1}{2R}\frac{\epsilon\mu - \eta^2}{\epsilon\mu - \eta^2 - \frac{n^2}{\omega^2 R^2}}.$$

One notices that the lower order term of the symbol does not change when we pass from the hyperbolic region to the elliptic region. The hyperbolic and elliptic region for the operator with respect to the boundary $r = R$ are classically defined. Note that, for all u, η, n, $|\sqrt{\epsilon\mu - \eta^2 - \frac{n^2}{\omega^2 r^2}}| \geq \sqrt{|\Im\epsilon\mu|}$ hence no glancing regime.

Theorem 2 *In the hyperbolic regime $\Re a(n,k) - \frac{1}{2}(\Re\epsilon\mu\omega^2 - k^2)\cosh 2u_0 < 0$, the asymptotics of $\frac{(F^a)'(n,k)(u_1)G^a(n,k)(u_0) - (G^a)'(n,k)(u_1)F^a(n,k)(u_0)}{F^a(n,k)(u_1)G^a(n,k)(u_0) - G^a(n,k)(u_1)F^a(n,k)(u_0)}$ read $i\omega\rho\sqrt{\frac{1}{2}(\epsilon\mu - \eta^2)\cosh 2u_1 - \frac{a(n,k)}{\rho^2\omega^2}}(1 + O(\omega^{-2}))$. The contribution of the curvature of $\partial\Omega$ appears through $\frac{1}{\rho\sqrt{\sin^2 v + \sinh^2 u_1}}$, and through the Floquet modes. It is non local in v.*

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