Oscillatory RBF for Helmholtz problems with large wavenumber

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Abstract

In this paper, we deal with numerical solutions of the Helmholtz equation using interpolation techniques through oscillatory radial basis functions (RBF). We build a radial basis function-generated finite differences (RBF-FD) scheme by means of Bessel functions as RBF. By means of regularization with diagonal increments we solve the ill-conditioned local interpolation problem. Finally, we test the solver with absorbing boundary conditions (ABC) and find numerical evidence showing that pollution effects are mitigated.

Keywords: RBF-FD, Helmholtz equation, shape parameter, pollution effect, oscillatory RBF, wave scattering.

1 Introduction

In this paper we calculate approximated numerical solutions for Helmholtz problems given by

$$\begin{cases} -\Delta u(\mathbf{x}) - \omega^2 c(\mathbf{x})^{-2} u(\mathbf{x}) &= f(\mathbf{x}), \text{ in } \Omega \\ b\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}) + \mathrm{i}\omega c(\mathbf{x})^{-1}\mathcal{B}u(\mathbf{x}) &= g(\mathbf{x}), \text{ on } \partial\Omega \end{cases} \tag{1}$$

where ω is the angular frequency, $c(\mathbf{x}) > 0$ is the sound speed of a continuous media $\Omega \subset \mathbb{R}^d$, $f(\mathbf{x})$ is the source term, $\mathbf{n}$ the is unitary normal vector field of the boundary $\partial\Omega$, b takes values zero or one, $\mathcal{B}$ is a certain linear operator[1] and $g(\mathbf{x})$ is certain exact data on $\partial\Omega$. Here $\mathrm{i} = \sqrt{-1}$.

Inspired by Trefftz methods [1], we use the class of oscillatory RBF given by

$$\varphi_k^{(d)}(r) = \frac{J_{d/2-1}(kr)}{(kr)^{d/2-1}}, \quad d = 1, 2, \dots, \tag{2}$$

where J_α is the Bessel function of the first kind. Since functions in (2) satisfy the homogeneous Helmholtz equation $\Delta u + k^2 u = 0$, any interpolant of the form

$$s(\mathbf{x}) = \sum_{j=1}^{n} \alpha_j \varphi_k^{(d)}(||\mathbf{x} - \mathbf{x}_j||)$$

will satisfy $\Delta s + k^2 s = 0$ as well. More details about this class of oscillatory functions can be consulted in [2].

2 Oscillatory RBF-FD for Helmholtz problems

We assume that u is the solution of the problem (1). With $X = \{\mathbf{x}_i\}_{i=1}^N \subset \Omega \cup \partial\Omega$ a set of nodes, we take stencils $S_i = \{\mathbf{x}_j^i\}_{j=1}^{n_i} \subset X$ based on $\mathbf{x}_i$, with $\mathbf{x}_1^i = \mathbf{x}_i$. For $\mathbf{x} \in \mathrm{ConvexHull}(S_i)$ we define, with $k_i = \omega^2 c(\mathbf{x}_i)^{-2}$, the interpolant

$$\widetilde{u}_i(\mathbf{x}) = \sum_{j=1}^{n_i} \alpha_j^i \varphi_{k_i}^{(d)}(\|\mathbf{x} - \mathbf{x}_j^i\|). \tag{3}$$

From (3), evaluating in the set of nodes we have the local interpolation matrix[2] for $\mathbf{x}_i$

$$\mathbf{J}_i = (\varphi_{k_i}^{(d)}(\|\mathbf{x}_l^i - \mathbf{x}_j^i\|))_{1 \le l,j \le n_i},$$

which is positive definite [2]. From (3) we have the linear equation $U_i = \mathbf{J}_i \boldsymbol{\alpha}_i$, where $U_i = \begin{pmatrix} \widetilde{u}_i(\mathbf{x}_1^i) & \widetilde{u}_i(\mathbf{x}_2^i) & \cdots & \widetilde{u}_i(\mathbf{x}_{n_i}^i) \end{pmatrix}^T$ and $\boldsymbol{\alpha}_i = \begin{pmatrix} \alpha_1^i & \alpha_2^i & \cdots & \alpha_{n_i}^i \end{pmatrix}^T$.

Using the interpolant[3] (3) in (1), we obtain the system of linear equations[4]

$$\begin{cases} \Gamma_i^T \mathbf{J}_i^{-1} U_i = f(\mathbf{x}_i), & \text{if } \mathbf{x}_i \in \Omega \\ \Theta_i^T \mathbf{J}_i^{-1} U_i = g(\mathbf{x}_i), & \text{if } \mathbf{x}_i \in \partial\Omega, \end{cases}$$

where

$$\Gamma_i = \left(\mathcal{F}\varphi_{k_i}^{(d)}(\|\mathbf{x} - \mathbf{x}_1^i\|)|_{\mathbf{x}=\mathbf{x}_i} \ \cdots \ \mathcal{F}\varphi_{k_i}^{(d)}(\|\mathbf{x} - \mathbf{x}_{n_i}^i\|)|_{\mathbf{x}=\mathbf{x}_i} \right),$$

$$\Theta_i = \left(\mathcal{G}\varphi_{k_i}^{(d)}(\|\mathbf{x} - \mathbf{x}_1^i\|)|_{\mathbf{x}=\mathbf{x}_i} \ \cdots \ \mathcal{G}\varphi_{k_i}^{(d)}(\|\mathbf{x} - \mathbf{x}_{n_i}^i\|)|_{\mathbf{x}=\mathbf{x}_i} \right),$$

$\mathcal{F} = -\Delta - \omega^2 c(\mathbf{x})^{-2}$, and $\mathcal{G} = b\frac{\partial}{\partial \mathbf{n}} + \mathrm{i}\omega c(\mathbf{x})^{-1}\mathcal{B}$.

[1]The operator $\mathcal{B}$, joint the value $b = 0, 1$, is used to define absorbing boundary conditions (ABC) of several orders, even thought to define Dirichlet or Neumann boundary conditions.

[2]Matrices $\mathbf{J}_i$ are often ill-conditioned and is necessary to use some regularization technical [3].

[3]Local interpolation is only used to calculate the weights to approximate the linear operators at each node, using the unknown values of the function u (similar to the finite difference method), which are the final representation calculated from the sparse system of linear equations.

[4]Which can be assembled in a sparse matrix equation.

$\frac{k}{2\pi}$	$\frac{1}{h}$	Nodes (N)	$\|u - \widetilde{u}\|_\infty$
10	60	3721	1.79e-04
20	120	14641	1.56e-04
40	240	58081	1.10e-04
80	480	231361	1.72e-04
120	720	519841	1.21e-04

Table 1: Results for approximated solutions of (3). With a square uniform grid in $\Omega \cap \partial\Omega$. For inner nodes the stencil size is $n = 13$, at boundary nodes $n_b = 15$, and the number of nodes per wavelength is kept constant with $N_g = 6$, i.e., $h = 2\pi/6k$.

3 Some examples and numerical results

In this section, we test the presented method with a couple of examples.

Test 1 With $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$, $k = \omega c^{-1}$, constant wave speed $c \equiv 1$, and

$$\begin{aligned} u(x,y) =& \sqrt{k}(h(-20,20) + 2h(20,20) \\ &+ 0.5h(-20,20) - h(20,-20)) \end{aligned}$$

with $h(x_0,y_0) = H_0^{(1)}(k\|\mathbf{x} - (x_0,y_0)\|)$, we solve

$$\begin{cases} -\Delta u(\mathbf{x}) - k^2 u(\mathbf{x}) &= 0, \quad \text{in } \Omega \\ \frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}) + \mathrm{i}k u(\mathbf{x}) &= g(\mathbf{x}), \ \text{on } \partial\Omega. \end{cases}$$

Results in Table 1 show that with a fixed number of nodes[5] per wavelength, the error keeps the same order as k increases, which is evidence that pollution effect is mitigated.

Test 2 We consider the acoustic scattering problem with a hard obstacle $\Omega_{obs} \subset \Omega$ and incident wave field given by $u_{inc}(\mathbf{x}) = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$, see Figure 1. The total wave field is given by $u = u_{inc} + u_{sct}$. To compute the scattered wave field we use the follow approximation by ABC on the boundary $\partial\Omega$

$$\begin{cases} -\Delta u_{sct}(\mathbf{x}) - \omega^2 u_{sct}(\mathbf{x}) = 0, & \text{in } \Omega/\overline{\Omega}_{obs} \\ u_{sct} = -u_{inc}, & \text{on } \partial\Omega_{obs} \\ \frac{\partial}{\partial \mathbf{n}} u_{sct}(\mathbf{x}) + \mathrm{i}\omega\mathcal{B}u_{sct}(\mathbf{x}) = 0, & \text{on } \partial\Omega \end{cases}$$

with $\mathcal{B} = 1 + \frac{1}{2}\omega^{-2}\frac{\partial^2}{\partial \boldsymbol{\tau}^2}$.

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[5]We know empirically that it is desirable to keep the symmetry of the stencils at the inner nodes and a larger size at the boundary nodes to reduce the error. About it, there is something related in [4].

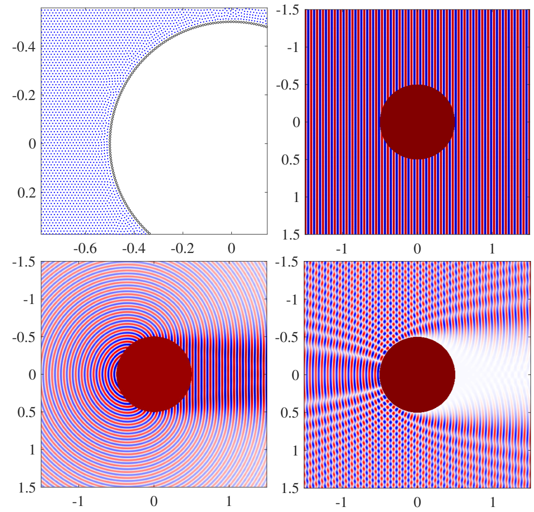

Figure 1: Top left: Distribution of nodes surrounding a circular obstacle Ω_{obs}. Top right: Incident wave field u_{inc}. Bottom left: Numerical solution of the scattered wave field u_{sct}. Bottom right: Total wave field $u = u_{inc} + u_{sct}$.

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