Stability analysis of the JCAPL equivalent fluid model equations for porous media

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Abstract

The equivalent fluid model (EFM) describes the acoustic properties of rigid porous media by defining it as a fluid with an effective density and an effective compressibility. Their definition are based on the dynamic tortuosity α and the dynamic compressibility β, known to be complex-valued functions depending on frequency. Among the different models describing α and β, this paper focuses on the Johnson-Champoux-Allard-Pride-Lafarge (JCAPL) model [1] where these parameters are defined as irrational functions, behaving like fractional derivatives in the time domain. Here, we present the proof of stability of the time-domain EFM using the JCAPL model thanks to their oscillatory-diffusive (OD) representations.

Keywords: porous media, equivalent fluid model, JCAPL model, OD representation, stability.

1 Introduction

The EFM equations for rigid porous materials are recalled below in the Laplace domain:

$$\begin{cases} \rho_0 \alpha(s)\ s\,\hat{\mathbf{u}} + \boldsymbol{\nabla} \hat{p} &= 0\,, \\ \chi_0 \beta(s)\ s\,\hat{p} + \boldsymbol{\nabla}\cdot\hat{\mathbf{u}} &= 0\,, \end{cases} \tag{1}$$

where ρ_0 is the ambient fluid density, χ_0 the ambient adiabatic compressibility; velocity $\mathbf{u}$ and pressure p are defined on $(0,\infty)\times\Omega$, with $\Omega \subset \mathbb{R}^n$, $\hat{f}$ denotes the Laplace transform of f and s is the complex variable.

The JCAPL model defines α and β as

$$\alpha(s) := \alpha_\infty \left[1 + \frac{M}{s} + N\frac{\sqrt{1+\frac{s}{L}}\text{-}1}{s}\right], \tag{2}$$

$$\beta(s) := \gamma\text{-}(\gamma\text{-}1)\left[1 + \frac{M'}{s} + N'\frac{\sqrt{1+\frac{s}{L'}}\text{-}1}{s}\right]^{-1}. \tag{3}$$

with parameters detailed in [2]. These expressions are based on the exact description of α and β at the high and low frequency limits, connected by a function whose singularities lie on the negative real axis of the complex plane [3].

In order to study the stability of the whole JCAPL - EFM system, a methodology based on a *poles and cuts* technique [4] is used. It enables to recast a complex function f in a formulation built from an OD representation (4), containing an infinite number of real or complex weights and poles, but no $\sqrt{s}$-type terms.

$$\hat{f}(s) = \underbrace{\sum_{k\in\mathbb{Z}} \frac{r_k}{s-s_k}}_{\text{oscillatory}} + \underbrace{\int_0^\infty \frac{\mu(\xi)}{s+\xi}\mathrm{d}\xi}_{\text{diffusive}}, \tag{4}$$

with the *well-posedness* condition $\int_0^\infty \frac{|\mu(\xi)|}{1+\xi}\,\mathrm{d}\xi < \infty$ for $\hat{f}$ to admit an OD representation [2].

2 Oscillatory-diffusive representation

The irrational parts of α and β are first studied in order to find their OD representation. The focus is therefore on the two following transfer functions:

$$\hat{g}(s) := \frac{1}{N}\left[\frac{\alpha(s)}{\alpha_\infty}\text{-}1\text{-}\frac{M}{s}\right] = \frac{\sqrt{1+\frac{s}{L}}\text{-}1}{s},$$

$$\hat{h}(s) := \frac{\beta(s)\text{-}1}{\gamma\text{-}1} = \frac{N'\sqrt{1+\frac{s}{L'}} + M'\text{-}N'}{s + N'\sqrt{1+\frac{s}{L'}} + M'\text{-}N'}.$$

Function $\hat{g}$ admits a diffusive representation with $\mu(\xi) \propto \xi^{-1}\,(\xi/L\text{-}1)^{1/2}$, a positive diffusive weight defined for $\xi \in (L,\infty)$ and verifying the well-posedness condition.

Function $\hat{h}$ has an oscillatory-diffusive representation containing a diffusive part and an additional isolated term, which is null for certain values of the parameters M', N' and L' [2].

$$\hat{h}(s) = \frac{r_0}{s-s_0} + \int_{L'}^\infty \frac{\nu(\xi)}{s+\xi}\mathrm{d}\xi, \tag{5}$$

with $s_0 < 0$, $r_0 \geqslant 0$ and $\nu(\xi) \propto \xi\,(\xi/L'\text{-}1)^{1/2} \left((\xi\text{-}M'\text{-}N')^2 + N'^2(\xi/L'\text{-}1)\right)^{-1}$, which is positive and verifies the well-posedness condition.

3 Extended diffusive realization

Based on the OD representations of $\hat{g}$ and $\hat{h}$, system (1) in the time domain reads:

$$\begin{cases} \partial_t \mathbf{u} + M\,\mathbf{u} + N\,(g\star\partial_t\mathbf{u}) + \dfrac{1}{\rho_0\,\alpha_\infty}\boldsymbol{\nabla}\,p = 0, \\ \partial_t p + (\gamma-1)\ (h\star\partial_t p) + \dfrac{1}{\chi_0}\boldsymbol{\nabla}\cdot\mathbf{u} = 0. \end{cases} \tag{6}$$

The interest of the OD representation is in the associated diffusive realization, which gives a time-local formulation of the convolution products present in (6). In this work, *extended* diffusive realizations, as

$$\begin{cases} \mathbf{z_u}(t,\mathbf{x}) := \displaystyle\int_L^\infty \mu(\xi)\,\partial_t \boldsymbol{\phi}(\xi;t,\mathbf{x})\,\mathrm{d}\xi\,, \\ \partial_t \boldsymbol{\phi}(\xi;t,\mathbf{x}) = -\,\xi\,\boldsymbol{\phi}(\xi;t,\mathbf{x}) \,+\, \mathbf{u}(t,\mathbf{x})\,, \\ \boldsymbol{\phi}(\xi;0,\mathbf{x}) = \mathbf{u}(0,\mathbf{x})/\xi\,, \end{cases} \tag{7}$$

for $\mathbf{z_u} = (g \star \partial_t \mathbf{u})$ with $\boldsymbol{\phi}$ the diffusive variable, are used and differ from *usual* diffusive realizations by the presence of a time derivative in the convolution product. Non-null initial conditions are set for the diffusive variable to have a finite value for $\mathbf{z_u}$ at $t=0$ [5]. The energy functional

$$E_{\boldsymbol{\phi}}(t) := \frac{1}{2}\int_\Omega \int_L^\infty \mu(\xi)\,\xi\,\|\boldsymbol{\phi}(\xi;t,\mathbf{x})\|^2\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x},$$

the derivative of which is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_{\boldsymbol{\phi}}(t) = \ & \int_\Omega \mathbf{u}(t,\mathbf{x})\cdot\mathbf{z_u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \,- \\ & \int_\Omega\int_L^\infty \mu(\xi)\,\|\partial_t\boldsymbol{\phi}(\xi;t,\mathbf{x})\|^2\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{8}$$

can be defined for the extended diffusive realization (7). A realization analogous to (7) is used for $z_p := (h \star \partial_t p)$ with ψ denoting its associated diffusive variable and E_ψ its associated energy. The additional first-order system $r_0/(s-s_0)$ in (5) is handled by the same diffusive variable ψ and its associated energy, included in E_ψ, is

$$E_{\psi_0}(t) := \frac{1}{2}\int_\Omega r_0\,(-s_0)\,|\psi(-s_0;t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x},$$

Using the realization of the convolution products in (6) leads to the augmented system:

$$\begin{cases} \partial_t \mathbf{u} + M\,\mathbf{u} + N\,\mathbf{z_u} + \dfrac{1}{\rho_0\,\alpha_\infty}\boldsymbol{\nabla}\,p = 0, \\ \partial_t p + (\gamma-1)\,z_p + \dfrac{1}{\chi_0}\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \\ \partial_t \boldsymbol{\phi}(\xi;t,\mathbf{x}) = -\,\xi\,\boldsymbol{\phi}(\xi;t,\mathbf{x}) \,+\, \mathbf{u}(t,\mathbf{x})\,, \\ \partial_t \psi(\xi;t,\mathbf{x}) = -\,\xi\,\psi(\xi;t,\mathbf{x}) \,+\, p(t,\mathbf{x})\,, \\ \boldsymbol{\phi}(\xi;0,\mathbf{x}) = \mathbf{u}(0,\mathbf{x})/\xi\,, \\ \psi(\xi;0,\mathbf{x}) = p(0,\mathbf{x})/\xi. \end{cases} \tag{9}$$

4 Stability analysis

The stability analysis of system (9) is performed thanks to the *augmented* energy functional

$$\mathcal{E}(t) := E_{\mathrm{m}}(t) + E_{\mathrm{diff}}(t),$$

where the classical mechanical energy is

$$E_{\mathrm{m}}(t) := \frac{\rho_0\alpha_\infty}{2}\int_\Omega \|\mathbf{u}\|^2\ \mathrm{d}x + \frac{\chi_0}{2}\int_\Omega |p|^2\ \mathrm{d}x,$$

and a *diffusive* energy is defined as

$$E_{\mathrm{diff}} := \rho_0\alpha_\infty N E_{\boldsymbol{\phi}}(t) + \chi_0(\gamma-1)E_\psi(t)\,.$$

The positivity of the JCAPL diffusive weights, and the known sign of r_0 and s_0, enables to prove the following proposition.

Theorem 4.1 *In a bounded domain Ω with no contribution at the boundary (either $p=0$, or $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$), the augmented energy $\mathcal{E}$ of the JCAPL-EFM satisties $\frac{d}{dt}\mathcal{E}(t) \leqslant 0$.*

The key point of the proof lies in noticing that certain terms in

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{m}}(t) = \ & -\rho_0\,\alpha_\infty\left(M\int_\Omega \|\mathbf{u}\|^2\ \mathrm{d}\mathbf{x} + N\int_\Omega \mathbf{u}\cdot\mathbf{z_u}\,\mathrm{d}\mathbf{x}\right) \\ & -\rho_0\,\alpha_\infty \,+\chi_0\int_\Omega p\,\partial_t p\,\mathrm{d}\mathbf{x}\,, \end{aligned}$$

have an opposite sign of those in the time derivative of the diffusive energy (see the first term of (8)) and can compensate exactly with $\dfrac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{diff}}$. Moreover, following [6], Prop. 4.1 can be proved.

Proposition 4.1 *The dynamical system (9) is asymptotically stable, i.e. $(\mathbf{u},p,\boldsymbol{\phi},\psi) \to (\mathbf{0},0,\mathbf{0},0)$ as $t\to\infty$ in the appropriate energy space.*

Hence, $\mathcal{E}$ describes an energy functional for (9), which enables to ensure the stability of the system without external inputs.

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