Multipole-model approximation of the equivalent fluid model equations for porous media in the time domain

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Abstract

The equivalent fluid model (EFM) describes the acoustic properties of rigid porous media by defining it as a fluid with an effective density ρ_{eq} and an effective compressibility C_{eq}. These physical quantities are complex-valued functions depending on frequency, and are irrational due to their behaviour at the high and low frequency limits. Hence, the system of equations derived from the EFM is first expressed in the Fourier domain, and can involve fractional derivatives in the time domain. Here, an approach is presented to formulate the EFM equations in the time domain, leading to a system with additional differential equations, on which an energy method is used to prove its stability under certain conditions.

Keywords: porous media, equivalent fluid model, multipole model, stability conditions.

1 Introduction

This work focuses on the equations of the EFM:

$$\left\{\begin{aligned} \rho_{eq}(s)\ s\,\hat{\mathbf{u}} + \boldsymbol{\nabla}\ \hat{p} &= 0\ , \\ C_{eq}(s)\ s\,\hat{p} + \boldsymbol{\nabla}\cdot\hat{\mathbf{u}} &= 0\ , \end{aligned}\right. \tag{1}$$

where $\mathbf{u}$ and p are defined on $(0,\infty)\times\Omega$, with $\Omega\subset\mathbb{R}^n$, $\hat{f}$ denotes the Laplace transform of f and s is the complex variable. In order to reformulate these equations in the time domain, recent studies have used the additional differential equations (ADE) method [1], approximating ρ_{eq} and C_{eq} by rational functions of s or using the oscillatory-diffusive (OD) representation method [2], based on a *pole and cut* technique [3].

Choosing to approximate ρ_{eq} and C_{eq} by rational functions or by using the OD representation approach leads, in both cases, to write their discrete expressions using multipole models.

$$\rho_{eq}(s) = \rho_\infty\left(1+\sum_{k=1}^{N}\frac{r_k}{s-s_k}\right), \tag{2}$$

$$C_{eq}(s) = C_\infty\left(1+\sum_{k=1}^{M}\frac{l_k}{s-q_k}\right). \tag{3}$$

On that account, the following work is based on ρ_{eq} and C_{eq} approximated as above.

2 Time-domain representation of the EFM

It is assumed here that the studied porous media have a purely dissipative behaviour, hence all parameters r_k, l_k, s_k and q_k of ρ_{eq} and C_{eq} are real. Then, using expressions (2) and (3), system (1) becomes:

$$\left\{\begin{aligned} &s\,\hat{\mathbf{u}} + \sum_{k=1}^{N}\frac{r_k}{s-s_k}s\,\hat{\mathbf{u}} + \frac{1}{\rho_\infty}\boldsymbol{\nabla}\ \hat{p} = 0\ , \\ &s\,\hat{p} + \sum_{k=1}^{M}\frac{l_k}{s-q_k}s\,\hat{p} + \frac{1}{C_\infty}\boldsymbol{\nabla}\cdot\hat{\mathbf{u}} = 0\ . \end{aligned}\right. \tag{4}$$

However, as suggested by Xie *et al.* for a Biot model [4], it can also be rewritten as follows, which can ease the numerical computation.

$$\left\{\begin{aligned} &s\,\hat{\mathbf{u}} + \sum_{k=1}^{N}\left(r_k + \frac{r_k s_k}{s-s_k}\right)\hat{\mathbf{u}} + \frac{1}{\rho_\infty}\boldsymbol{\nabla}\ \hat{p} = 0\ , \\ &s\,\hat{p} + \sum_{k=1}^{M}\left(l_k + \frac{l_k q_k}{s-q_k}\right)\hat{p} + \frac{1}{C_\infty}\boldsymbol{\nabla}\cdot\hat{\mathbf{u}} = 0\ . \end{aligned}\right. \tag{5}$$

In order to compare these two equivalent systems in the time domain, the inverse Laplace transform is applied. This leads to a new set of equations containing causal convolution. To compute them, additional variables, $\boldsymbol{\varphi_k}$ and ψ_k for system (4), $\boldsymbol{\varphi_k^\star}$ and $\psi_k^\star$ for system (5), are introduced. Therefore, the global systems for (4) and (5) read:

$$\left\{\begin{aligned} &\partial_t\mathbf{u} + \frac{1}{\rho_\infty}\boldsymbol{\nabla}\ p = -\sum_{k=1}^{N} r_k\boldsymbol{\varphi_k}\ , \\ &\partial_t p + \frac{1}{C_\infty}\boldsymbol{\nabla}\cdot\mathbf{u} = -\sum_{k=1}^{M} l_k\psi_k\ , \\ &\partial_t\boldsymbol{\varphi_k} + \frac{1}{\rho_\infty}\boldsymbol{\nabla}\ p = s_k\boldsymbol{\varphi_k} - \sum_{k=1}^{N} r_k\boldsymbol{\varphi_k}\ , \\ &\partial_t\psi_k + \frac{1}{C_\infty}\boldsymbol{\nabla}\cdot\mathbf{u} = q_k\psi_k - \sum_{k=1}^{M} l_k\psi_k\ ; \end{aligned}\right. \tag{6}$$

$$\begin{cases} \partial_t \mathbf{u} + \dfrac{1}{\rho_\infty} \boldsymbol{\nabla}\ p = -\displaystyle\sum_{k=1}^{N} r_k \mathbf{u} - \sum_{k=1}^{N} r_k s_k \boldsymbol{\varphi}^\star_{\boldsymbol{k}}\ , \\ \partial_t p + \dfrac{1}{C_\infty} \boldsymbol{\nabla} \cdot \mathbf{u} = -\displaystyle\sum_{k=1}^{M} l_k p - \sum_{k=1}^{M} l_k q_k \psi^\star_k\ , \\ \partial_t \boldsymbol{\varphi}^\star_{\boldsymbol{k}} = s_k \boldsymbol{\varphi}^\star_{\boldsymbol{k}} + \mathbf{u}\ , \\ \partial_t \psi^\star_k = q_k \psi^\star_k + p\ . \end{cases} \quad (7)$$

In (7), there are no spatial derivatives in the ADE, compared to (6). Hence, when the system is discretized with a numerical scheme based on fluxes, these fluxes depend on the velocity and pressure variables, but not on the additional variables. Consequently, the problem to solve at each mesh interface does not grow with the number of additional variables. Moreover, for problems with multiple subdomains (porous media and air domain), there are no additional fluxes to manage for the interface between them.

3 Stability analysis

Hereafter, a stability analysis of system (7) is performed thanks to an energy functional defined below:

$$\mathcal{E}(t) = \rho_\infty \mathcal{E}_u(t) + C_\infty \mathcal{E}_p(t)\ ,$$

where $\mathcal{E}_u = \dfrac{1}{2}\displaystyle\int_\Omega \|\mathbf{u}\|^2\ \mathrm{d}x \text{ - } \sum_{k=1}^{N} \frac{r_k s_k}{2} \int_\Omega \|\boldsymbol{\varphi}^\star_{\boldsymbol{k}}\|^2\ \mathrm{d}x\ ,$

and $\quad \mathcal{E}_p = \dfrac{1}{2}\displaystyle\int_\Omega p^2\, \mathrm{d}x \text{ - } \sum_{k=1}^{M} \frac{l_k q_k}{2} \int_\Omega (\psi^\star_k)^2\ \mathrm{d}x\ .$

It is straightforward to take its derivative:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \quad & -\rho_\infty \sum_{k=1}^{N} r_k \int_\Omega \|\partial_t \boldsymbol{\varphi}^\star_{\boldsymbol{k}}\|^2\ \ \mathrm{d}x \\ & -C_\infty \sum_{k=1}^{M} l_k \int_\Omega (\partial_t \psi^\star_k)^2\ \ \mathrm{d}x \\ & -\int_{\partial\Omega} (p\,\mathbf{u}) \cdot \mathbf{n}\ \mathrm{d}\sigma. \end{aligned}$$

From these equations, combined with the fact that ρ_∞ and C_∞ are necessarily positive, we impose the following sufficient conditions on the parameters:

($\mathbf{C_1}$) the weights $(r_k, l_k)_k$ are positive,
($\mathbf{C_2}$) the poles $(s_k, q_k)_k$ are negative.

Hence, $\mathcal{E}$ is positive-definite and decreasing without external contributions at the boundary of the domain Ω. Thus, under these conditions, $\mathcal{E}$ describes an energy functional for (7), that enables to ensure the stability of the system without external inputs.

A numerical application with a Discontinuous Galerkin (DG) scheme was carried out to apply previous results.

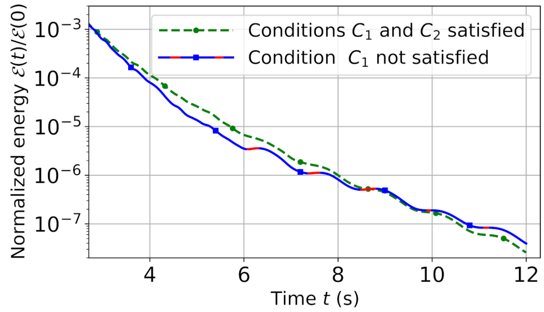

Figure 1: Normalized energy over time in a 2D domain for different multipole-model parameters approximating a toy model and an initial Gaussian field for $\mathbf{u}$ and p. Increasing parts of the energy are marked in red.

Furthermore, these 2 conditions are met by several approximated models describing ρ_{eq} and C_{eq} for a realistic range of physical parameters. Indeed, the OD representation of the JCAPL model or the Horoshenkov model can be discretized in the form of (2) and (3) with parameters verifying the conditions ($\mathbf{C_1}$) and ($\mathbf{C_2}$).

4 Conclusion

This work formulates the EFM equations for porous media in the time domain, with the functions ρ_{eq} and C_{eq} described by a set of weights and poles. In addition, a proof of stability is given under certain conditions on the sign of these weights and poles. In order to apply the theoretical results obtained, a DG scheme was implemented in 2D [5].

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