Stability of space–time isogeometric methods for wave propagation problems

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Abstract

We investigate the first steps towards an unconditionally stable space–time isogeometric (IGA) discretization for the second-order wave equation. Inspired by a finite element formulation proposed by Steinbach and Zank, we propose a stabilization of the isogeometric method for an ordinary differential equation that is closely related to the wave equation. This suggests an extension to wave propagation problems.

Keywords: Isogeometric Analysis, wave equation, space–time Galerkin formulation, stability.

1 Introduction

The space–time discretization of evolution equations is a fairly recent tool that offers approximate solutions that are available at all times in the interval of interest, in contrast to semi-discretization and time-stepping techniques.

We focus on the following scalar, second-order wave-propagation problem with homogeneous conditions

$$\begin{cases} \partial_{tt}u - \Delta u = g & \text{on } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = \partial_t u = 0 & \text{on } \Omega \times \{0\}, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded, Lipschitz domain. A variational formulation of (1) with integration by parts in both space and time is considered in [2]. A CFL condition $h_t \leq C h_x$ is required for the stability of conforming tensor-product space–time discretizations with piecewise-linear polynomials. Different approaches have been proposed in order to overcome the CFL condition. In [1] the stability of the conforming piecewise-linear FEM is addressed by first studying the same discretization applied to the Helmholtz ODE initial value problem

$$\begin{cases} \partial_{tt}u(t) + \mu u(t) = f(t) \quad t \in (0,T), \\ u(0) = (\partial_t u)(0) = 0 \end{cases} \tag{2}$$

with $\mu > 0$. Linear-FEM unconditional stability and optimal convergence rates in space–time norms are proved for both (1) and (2) using a perturbed variational formulation.

Motivated by the excellent numerical properties of the IGA method, we aim at extending the techniques of [1] to high-order space–time isogeometric discretizations.

2 Abstract variational problem

As in [1], our model problem is the ODE (2).

Define the following subspaces of $H^1(0,T)$:

$$H^1_{0,*}(0,T) = \{w \in H^1(0,T) : \; w(0) = 0\},$$
$$H^1_{*,0}(0,T) = \{v \in H^1(0,T) : \; v(T) = 0\},$$

endowed with the Sobolev seminorm $|\cdot|_{H^1(0,T)}$.

The variational formulation of (2) reads

$$\begin{cases} \text{Find } u \in H^1_{0,*}(0,T) \quad \text{such that} \\ a(u,v) = \langle f, v\rangle_{(0,T)} \quad \forall v \in H^1_{*,0}(0,T), \end{cases} \tag{3}$$

where $T > 0$ and $f \in [H^1_{*,0}(0,T)]'$ are given, and where the bilinear form is

$$a(\cdot,\cdot) : H^1_{0,*}(0,T) \times H^1_{*,0}(0,T) \longrightarrow \mathbb{R}$$
$$a(w,v) := -\langle \partial_t w, \partial_t v\rangle_{L^2(0,T)} + \mu\langle w, v\rangle_{L^2(0,T)}.$$

As proven in [3], problem (3) is well-posed.

3 Isogeometric discretization

As discrete trial and test spaces for a conforming Galerkin discretization of (3) we consider

$$V^h_{0,*} := S^p_h(0,T) \cap H^1_{0,*}(0,T),$$
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where $S^p_h(0,T)$ is the space of degree-p piecewise-polynomials in $C^{p-1}(0,T)$ (i.e. maximal-regularity splines), and h is the mesh size. The conforming Petrov–Galerkin isogeometric discretization of (3) is

$$\begin{cases} \text{Find } u_h \in V^h_{0,*} \quad \text{such that} \\ a(u_h,v_h) = \langle f, v_h\rangle_{(0,T)} \quad \forall v_h \in V^h_{*,0}. \end{cases} \tag{4}$$

By extending Theorem 4.7 of [2] (in the case $p = 2$) and by the use of compact perturbations

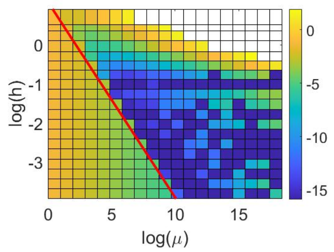

Figure 1: The inf-sup constant of quadratic IGA in log scale. The red line is condition (5).

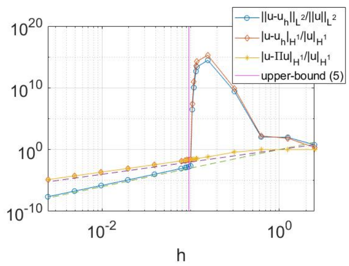

Figure 2: Relative errors of the non-stabilized IGA discretization (4) for $p = 2$. The exact solution is $u(t) = \sin^2\left(\frac{5}{4}\pi t\right)$, for $t \in (0, 10)$ and $\mu = 1000$.

techniques (for any p), we get two results of conditional (w.r.t. h) stability for (4). However, numerical results show that these two constraints are not sharp; their improvement is currently ongoing. Our numerical experiments also suggest that, if the mesh size satisfies

$$h < \sqrt{\frac{9}{\mu}}, \tag{5}$$

then the quadratic isogeometric discretization (4) is inf-sup stable; see Figures 1–2.

The stabilized bilinear form in [1, (17.13)] can easily be written as

$$a_h(w_h, v_h) = -\langle \partial_t w_h, \partial_t v_h \rangle_{L^2(0,T)}$$
$$+ \mu \langle w_h, v_h \rangle_{L^2(0,T)} - \frac{\mu}{12} \sum_{l=1}^{N_F} h_l^2 \langle \partial_t w_h, \partial_t v_h \rangle_{L^2(\tau_l)},$$

where $\{\tau_l\}_{l=1,\dots,N_F}$ are the elements of the FEM mesh. We thus propose to substitute in place of $a(\cdot,\cdot)$ in (4) the discrete bilinear form

$$a_h(w_h, v_h) := -\langle \partial_t w_h, \partial_t v_h \rangle_{L^2(0,T)} \tag{6}$$
$$+ \mu \langle w_h, v_h \rangle_{L^2(0,T)} - \delta_p \mu \sum_{l=1}^{N_I} h_l^{2p} \langle \partial_t^p w_h, \partial_t^p v_h \rangle_{L^2(\tau_l)},$$

where $\delta_p > 0$ is a penalty parameter. The effect of this stabilization for $p = 2$ and $\delta_p = \frac{1}{100}$ is visible in Figures 3–4, where we can observe the enlargement of the stable region that corresponds to inf-sup values larger than $\approx e^{-5}$.

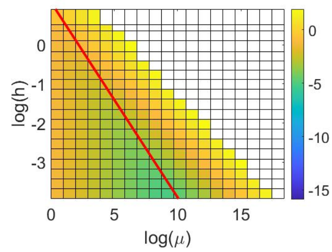

Figure 3: The inf-sup constant of the perturbed quadratic IGA (6). The red line is condition (5).

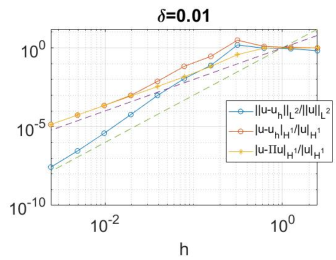

Figure 4: Relative errors of the perturbed IGA discretization (6) for $p = 2$. We see quadratic convergence in $|\cdot|_{H^1(0,T)}$ and cubic in $\|\cdot\|_{L^2(0,T)}$.

In particular, the inf-sup constant is stable for $h \to 0$ and fixed μ.

4 Wave equation

Numerical tests show that a conforming isogeometric discretization of the space–time variational formulation of (1), as introduced in [2], requires a CFL condition. Following Section 3, we propose to replace the bilinear form of the space–time IGA scheme with

$$-\langle \partial_t w_h, \partial_t v_h \rangle_{L^2(Q)} + \langle \nabla_x w_h, \nabla_x v_h \rangle_{L^2(Q)} \tag{7}$$
$$-\delta_p^W \sum_{m=1}^{d} \sum_{l=1}^{N_t} h_l^{2p} \langle \partial_t^p \partial_{x_m} w_h, \partial_t^p \partial_{x_m} v_h \rangle_{L^2(\Omega\times\tau_l)}$$

for $\delta_p^W > 0$. Numerical tests are ongoing.

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