A space–time quasi-Trefftz DG method for the wave equation with smooth coefficients

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Abstract

We propose a quasi-Trefftz method to approximate initial boundary value problems for the acoustic wave equation with piecewise-smooth material parameters. The key feature of the scheme is that all discrete trial and test functions are elementwise approximate solutions of the wave equation. The quasi-Trefftz scheme is framed in a space–time discontinuous Galerkin (DG) setting. We prove stability and high-order convergence, and show that the number of DOFs needed to obtain a given accuracy is considerably smaller than for schemes based on classical polynomial spaces. The quasi-Trefftz basis functions are polynomials in the space–time variable and can be computed with a simple algorithm. The inspiration comes from the generalised plane waves developed for time-harmonic problems with variable coefficients.

Keywords: quasi-Trefftz, space–time, discontinuous Galerkin, wave equation.

1 Trefftz and quasi-Trefftz methods

Trefftz schemes are Galerkin methods whose test and trial spaces are made of elementwise solutions of the PDE to be approximated. They are well-studied for homogeneous, linear PDEs with piecewise-constant coefficients, see [3] for the case of the wave equation. Other examples of Trefftz schemes use harmonic polynomials for the Laplace equation $\Delta u = 0$ and plane waves for the Helmholtz equation $\Delta u + k^2 u = 0$.

When the equation coefficients are not constant, it is usually very difficult to construct families of exact solutions with good approximation properties to use as basis functions, so Trefftz schemes are not viable in this case.

Quasi-Trefftz methods use discrete spaces of functions that are approximate solutions of the PDE. With this, we mean that the Taylor polynomial (of some given order m) of $\mathcal{L}v_h$, $\mathcal{L}$ being the PDE operator and v_h any discrete function, vanishes in a given point of each element of the computational mesh. In this way it is possible to construct low-dimensional discrete spaces with excellent approximation properties.

Existing quasi-Trefftz methods for time-harmonic problems use exponential basis functions called "generalized plane waves", see [1]. Here, instead, we develop a quasi-Trefftz discretisation for the time-domain scalar wave equation with variable coefficients. Building on the Trefftz case studied in [3], we define space–time quasi-Trefftz polynomial bases, we show how to compute them and we use them in a DG scheme.

All details can be found in [2].

2 Variable-coefficient wave equation

We consider the following initial boundary value problem for the first-order system corresponding to the homogeneous acoustic wave equation:

$$\begin{cases} \nabla v + \rho \partial_t \boldsymbol{\sigma} = \mathbf{0} & \text{in } Q = \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{\sigma} + G \partial_t v = 0 & \text{in } Q, \\ v(\cdot, 0) = v_0, \ \boldsymbol{\sigma}(\cdot, 0) = \boldsymbol{\sigma}_0 & \text{on } \Omega \subset \mathbb{R}^n, \\ v = g_D & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is an open, bounded, Lipschitz polytope, $\rho, G > 0$ are the material coefficients, independent of time and piecewise-smooth, and $c := (\rho G)^{-1/2}$ is the wavespeed. Neumann ($\boldsymbol{\sigma} \cdot \mathbf{n_x} = g_N$) and Robin ($\vartheta v - \boldsymbol{\sigma} \cdot \mathbf{n_x} = g_R$) boundary conditions may also be included.

If $\rho \boldsymbol{\sigma}_0$ is a gradient, then $v = \partial_t u$ and $\boldsymbol{\sigma} = -\frac{1}{\rho}\nabla u$, where u is a solution of the second-order, scalar, homogeneous wave equation

$$\Box_{\rho,G} u := -\nabla \cdot \left(\frac{1}{\rho}\nabla u\right) + G\partial_t^2 u = 0 \quad \text{in } Q.$$

3 Local polynomial quasi-Trefftz spaces

Let $K \subset Q$ be a space–time subdomain (a mesh element) that is star-shaped with respect to a centre point $(\mathbf{x}_K, t_K) \in K$. Assume that the material parameters ρ, G are sufficiently smooth in K. We define the local quasi-Trefftz spaces

$$\mathbb{QU}^p(K) := \big\{ f \in \mathbb{P}^p(K) \mid D^{\mathbf{i}} \square_{\rho,G} f(\mathbf{x}_K, t_K) = 0, \\ \forall \mathbf{i} \in \mathbb{N}_0^{n+1},\ |\mathbf{i}| \le p-2 \big\}, \quad p \in \mathbb{N},$$

$$\mathbb{QW}^p(K) := \Big\{ \Big(\partial_t f, -\frac{1}{\rho} \nabla f \Big),\ f \in \mathbb{QU}^{p+1}(K) \Big\},$$

where $\mathbb{P}^p(K)$ is the space of polynomials of degree at most p on K, and $D^{\mathbf{i}}$, with multi-index $\mathbf{i} = (\mathbf{i}_{\mathbf{x}}, i_t) = (i_1, \ldots, i_n, i_t)$, denotes the space–time partial derivative $D^{\mathbf{i}} = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \partial_t^{i_t}$.

$\mathbb{QU}^p(K)$ is the space of polynomials f of degree at most p such that the degree-$(p-2)$ Taylor polynomial of $\square_{\rho,G} f$ at $(\mathbf{x}_K, t_K)$ is zero. Similary, $\mathbb{QW}^p(K)$ is the analogous space for the first-order system. It is possible to define a third kind of spaces $\mathbb{QT}^p(K)$ for first-order systems not coming from second-order equations.

For all smooth solutions u of $\square_{\rho,G} u = 0$ in K, the degree-p Taylor polynomial $T[u](\mathbf{x}, t) = \sum_{|\mathbf{i}| \le p} \frac{(\mathbf{x}-\mathbf{x}_K)^{\mathbf{i}_{\mathbf{x}}} (t-t_K)^{i_t}}{\mathbf{i}!} D^{\mathbf{i}} u(\mathbf{x}, t)$ belongs to the space $\mathbb{QU}^p(K)$. It follows that for all $0 \le q \le p$

$$\inf_{P \in \mathbb{QU}^p(K)} |u - P|_{C^q(K)} \le C_{p,q,n} r_K^{p+1-q} |u|_{C^{p+1}(K)},$$

where $r_K := \sup_{(\mathbf{x},t) \in K} |(\mathbf{x}, t) - (\mathbf{x}_K, t_K)|$. This means that the quasi-Trefftz space $\mathbb{QU}^p(K)$ approximates all $C^{p+1}(K)$ PDE solutions with the same h-convergence rates of the full polynomial space $\mathbb{P}^p(K)$. The advantage of the quasi-Trefftz space is that it is a much smaller space:

$$\dim\big(\mathbb{QU}^p(K)\big) = \mathcal{O}_{p\to\infty}(p^n) \\ \ll \dim\big(\mathbb{P}^p(K)\big) = \mathcal{O}_{p\to\infty}(p^{n+1}).$$

4 Basis function construction

We want to define a concrete basis of the quasi-Trefftz space. We note that the coefficients $a_{\mathbf{i}}$ of the monomial expansion $v(\mathbf{x}, t) = \sum_{|\mathbf{i}| \le p} a_{\mathbf{i}} (\mathbf{x} - \mathbf{x}_K)^{\mathbf{i}_{\mathbf{x}}} (t - t_K)^{i_t}$ of any quasi-Trefftz polynomial $v \in \mathbb{QU}^p(K)$ satisfy the recurrence relations

$$a_{\mathbf{i}_{\mathbf{x}}, i_t+2} = -\sum_{\mathbf{j}_{\mathbf{x}} < \mathbf{i}_{\mathbf{x}}} \frac{g_{\mathbf{i}_{\mathbf{x}} - \mathbf{j}_{\mathbf{x}}}}{g_{\mathbf{0}} a_{\mathbf{j}_{\mathbf{x}}, i_t+2}} \\ + \sum_{l=1}^{n} \sum_{\mathbf{j}_{\mathbf{x}} \le \mathbf{i}_{\mathbf{x}} + \mathbf{e}_l} \frac{(i_{x_l}+1)(j_{x_l}+1)\, \zeta_{\mathbf{i}_{\mathbf{x}} + \mathbf{e}_l - \mathbf{j}_{\mathbf{x}}}}{(i_t+2)(i_t+1)\, g_{\mathbf{0}}} a_{\mathbf{j}_{\mathbf{x}} + \mathbf{e}_l, i_t},$$

where $g_{\mathbf{i}}$ and $\zeta_{\mathbf{i}}$ are the Taylor coefficients of G and ρ^{-1} at $(\mathbf{x}_K, t_K)$. It is possible to order these relations in such a way that all coefficients of v can be computed from $a_{\mathbf{i}_{\mathbf{x}},0}$ and $a_{\mathbf{i}_{\mathbf{x}},1}$. It follows that v is determined by its value and the value of $\partial_t v$ at time $t = t_K$.

This implies that, given a basis $\widehat{\mathcal{B}}$ of $\mathbb{P}^p(\mathbb{R}^n)$ and a basis $\widetilde{\mathcal{B}}$ of $\mathbb{P}^{p-1}(\mathbb{R}^n)$ (these are polynomials in $\mathbf{x}$ only), we can construct a basis $\mathcal{B}$ of $\mathbb{QU}^p(K)$ such that each $b_J \in \mathcal{B}$ satisfies either

$$\begin{cases} b_J(\cdot, t_K) \in \widehat{\mathcal{B}}, \\ \partial_t b_J(\cdot, t_K) = 0, \end{cases} \quad \text{or} \quad \begin{cases} b_J(\cdot, t_K) = 0, \\ \partial_t b_J(\cdot, t_K) \in \widetilde{\mathcal{B}}. \end{cases}$$

Then the relations above allow to explicitly compute the coefficients $a_{\mathbf{i}}$ of the monomial expansion of b_J with a simple iterative algorithm.

5 Quasi-Trefftz DG method

Let $\mathcal{T}_h$ be a polytopic mesh that partitions the space–time cylinder Q. The global quasi-Trefftz space $\prod_{K \in \mathcal{T}_h} \mathbb{QW}^p(K)$ is used as trial and test discrete space of a DG method that extends the one introduced in [3]. The formulation employs centred-in-space and upwind-in-time numerical fluxes on interior mesh faces.

In [2] we show that, under appropriate assumptions, the DG scheme is well-posed and we prove high-order h-convergence rates. These are optimal in a skeleton norm and half-order suboptimal at final time (i.e. in $L^2(\Omega \times \{T\})$ norm).

Numerical examples validate the theoretical results for both Cartesian-product and tent-pitched space–time meshes. These meshes give rise to a sort of implicit and explicit advancement in time, respectively. The main advantage compared to standard DG schemes is the faster convergence in terms of the number of degrees of freedom.

References

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