Computing singular and near-singular integrals in high-order boundary elements

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Abstract

We present in this talk algorithms for computing singular and near-singular integrals arising when solving the 3D Helmholtz equation with high-order boundary elements. These are based on the computation of the preimage of the singularity on the reference element using Newton's method, singularity subtraction with high-order Taylor-like asymptotic expansions, the continuation approach, and transplanted Gauss quadrature. We demonstrate the accuracy with several numerical experiments, including the scattering by two nearby half-spheres.

Keywords: Helmholtz equation, integral equations, boundary element method, singular integrals, continuation approach, Gauss quadrature

1 Introduction

The Helmholtz equation $\Delta u + k^2 u = 0$ in the presence of an obstacle may be rewritten as an integral equation on the obstacle's boundary via *layer potentials*. For example, the radiating solution to the Dirichlet problem $\Delta u + k^2 u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ with $u = u_D$ on $\Gamma = \partial\Omega$, for some bounded Ω whose complement is connected, can be obtained via the equation

$$\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})d\Gamma(\boldsymbol{y}) = u_D(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma, \quad (1)$$

based on the *single-layer potential*; the function G is the Green's function

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|}. \quad (2)$$

Once (1) is solved for φ, which is unique if k^2 is not an eigenvalue of $-\Delta$ in Ω, the solution u may be represented by the left-hand side of (1) for all $\boldsymbol{x}\in\mathbb{R}^3\setminus\Omega$.

From a numerical point of view, integral equations of the form of (1) are particularly challenging for several reasons. First, when $\boldsymbol{x}$ approaches $\boldsymbol{y}$, the integral becomes singular and standard quadrature schemes fail to be accurate—analytic integration or carefully-derived quadrature formulas are, hence, required. Second, the resulting linear systems after discretization are often dense. For large wavenumbers k, only iterative methods can be used to solve them (with the help of specialized techniques to accelerate the matrix-vector products, such as the Fast Multipole Method or hierarchical matrices). In this respect, the use of higher-order numerical discretization schemes may be helpful in enlarging the interval of feasible wavenumbers.

2 Computing singular integrals

When solving (1) with high-order boundary elements, one has to compute integrals of the form

$$I(\boldsymbol{x}_0) = \int_{\mathcal{T}} \frac{\varphi(F^{-1}(\boldsymbol{x}))}{|\boldsymbol{x}-\boldsymbol{x}_0|}dS(\boldsymbol{x}),$$

where $\mathcal{T}$ is a curved triangular element defined by a polynomial transformation $F:\widehat{T}\to\mathcal{T}$ of degree $q\geq 1$ from the flat reference triangle $\widehat{T}$, $\boldsymbol{x}_0$ is a point on or close to the element $\mathcal{T}$, and $\varphi:\widehat{T}\to\mathbb{R}$ is of degree $p\geq 0$. Our method goes like this.

Step 1. We map $\mathcal{T}$ back to $\widehat{T}$,

$$I(\boldsymbol{x}_0) = \int_{\widehat{T}} \frac{\psi(\hat{\boldsymbol{x}})}{|F(\hat{\boldsymbol{x}})-\boldsymbol{x}_0|}dS(\hat{\boldsymbol{x}}),$$

where $F:\widehat{T}\to\mathcal{T}$, the transformation of degree q from 2D flat $\widehat{T}$ to 3D curved $\mathcal{T}$, has a 3×2 Jacobian matrix J with columns J_1 and J_2, and $\psi(\hat{\boldsymbol{x}}) = \varphi(\hat{\boldsymbol{x}})|J_1(\hat{\boldsymbol{x}})\times J_2(\hat{\boldsymbol{x}})|$.

Step 2. We write

$$\boldsymbol{x}_0 = F(\hat{\boldsymbol{x}}_0) + \boldsymbol{x}_0 - F(\hat{\boldsymbol{x}}_0)$$

for some $\hat{\boldsymbol{x}}_0\in\widehat{T}$ such that $F(\hat{\boldsymbol{x}}_0)\in\mathcal{T}$ is the closest point to $\boldsymbol{x}_0$ on $\mathcal{T}$. We introduce here the parameter of singularity $h = |F(\hat{\boldsymbol{x}}_0)-\boldsymbol{x}_0|$.

Step 3. We compute the singular term using the Taylor series of $F(\hat{\boldsymbol{x}})-\boldsymbol{x}_0$ at $\hat{\boldsymbol{x}}_0$,

$$T_{-1}(\hat{\boldsymbol{x}},h) = \frac{\psi(\hat{\boldsymbol{x}}_0)}{\sqrt{|J(\hat{\boldsymbol{x}}_0)(\hat{\boldsymbol{x}}-\hat{\boldsymbol{x}}_0)|^2+h^2}},$$

and add it to/subtract it from $I(\boldsymbol{x}_0)$, *i.e.*,

$$I(\boldsymbol{x}_0) = I_{-1}(h) \\ + \int_{\widehat{T}} \left[\frac{\psi(\hat{\boldsymbol{x}})}{|F(\hat{\boldsymbol{x}}) - \boldsymbol{x}_0|} - T_{-1}(\hat{\boldsymbol{x}}, h) \right] dS(\hat{\boldsymbol{x}}).$$

The singular integral $I_{-1}(h)$ is given by

$$I_{-1}(h) = \int_{\widehat{T}} T_{-1}(\hat{\boldsymbol{x}}, h) dS(\hat{\boldsymbol{x}}),$$

and will be computed in Steps 4–5. The other integral has a bounded integrand—it can be computed with standard Gauss quadrature.

Step 4. The integrand in $I_{-1}(h)$ is homogeneous in both $\hat{\boldsymbol{x}}$ and h, and using the continuation approach, we reduce the 2D integral to a sum of three 1D integrals along the edges of the shifted triangle $\widehat{T} - \hat{\boldsymbol{x}}_0$,

$$I_{-1}(h) = \psi(\hat{\boldsymbol{x}}_0) \sum_{j=1}^{3} \hat{s}_j \int_{\partial\widehat{T}_j - \hat{\boldsymbol{x}}_0} g_h(\hat{\boldsymbol{x}}) ds(\hat{\boldsymbol{x}}),$$

with 1D smooth functions

$$g_h(\hat{\boldsymbol{x}}) = \frac{\sqrt{|J(\hat{\boldsymbol{x}}_0)\hat{\boldsymbol{x}}|^2 + h^2} - h}{|J(\hat{\boldsymbol{x}}_0)\hat{\boldsymbol{x}}|^2}.$$

The $\hat{s}_j$'s are the distances from the origin to the edges of the shifted triangle.

Step 5. On the one hand, when the origin is far from all three edges, each integrand is analytic—convergence with Gauss quadrature is exponential. On the other, when the origin lies on an edge, the corresponding integrand is singular—however, the distance to that edge is 0, the product "$\hat{s}_j$ times integral" is also 0, and the integral does not need to be computed at all. Issues arise when the origin is close to one of the edges—the integrand is analytic but near-singular, convergence with Gauss quadrature is exponential but slow. We circumvent this issue by using transplanted Gauss quadrature.