Stable finite differences for the piecewise homogeneous dynamic beam equation

Gustav Eriksson[1,*], Jonatan Werpers, David Niemelä, Niklas Wik, Valter Zethrin, Ken Mattsson

[1]Department of Information Technology, Uppsala University, Uppsala, Sweden

*Email: gustav.eriksson@it.uu.se

Abstract

The imposition of interface conditions for the piecewise homogeneous dynamic beam equation (DBE) is considered. Based on high order summation-by-parts finite differences, two novel energy stable methods are presented. Numerical experiments comparing the methods verify the theoretical convergence expectations and show that both methods are similar in terms of accuracy.

Keywords: dynamic beam equation, interface treatment, summation by parts, finite differences

1 Introduction

The dynamic beam equation (DBE) is a standard beam theory model describing the motion of free vibrations of a Euler-Bernoulli beam. Today it is used in construction of infrastructure involving beams such as buildings, bridges and railways.

The DBE is a dispersive wave equation for which the group velocity depends linearly on the wave number [1]. Consequently, the time step of any numerical method must scale as the square of the spatial step to resolve high frequency components of the solution. To keep the spatial step size small and still obtain an accurate solution, a high order finite difference method with summation-by-parts (SBP) properties is suggested. Together with either the projection method (SBP-P) or simultaneous approximation terms (SBP-SAT) to impose the interface conditions, the resulting numerical schemes can be proven stable using the energy method.

In this short paper novel SBP-P and SBP-SAT discretizations are presented and compared for the DBE with discontinuous material parameters. To save space, the proofs of energy conservation for the continuous and the two semi-discrete problems are omitted.

2 Continuous problem

The dynamic beam equation with a material discontinuity at $x = 0$ is given by

$$\begin{aligned} b^{(1)}u_{tt} &= -a^{(1)}u_{xxxx}, \quad x \in [-1, 0], \\ b^{(2)}v_{tt} &= -a^{(2)}v_{xxxx}, \quad x \in [0, 1], \end{aligned} \tag{1}$$

where $a^{(1,2)}$ and $b^{(1,2)}$ are positive constants incorporating the material parameters in each block and u and v denote the solutions in each block. The interface conditions at $x = 0$ ensuring energy conservation are given by

$$\begin{aligned} u &= v, \\ u_x &= v_x, \\ a^{(1)}u_{xx} &= a^{(2)}v_{xx}, \\ a^{(1)}u_{xxx} &= a^{(2)}v_{xxx}. \end{aligned} \tag{2}$$

3 Semi-discrete problem

The two blocks $[-1, 0]$ and $[0, 1]$ are discretized into equidistant grids with m grid points and step size h. Let u and v denote the semi-discrete solution vectors. The spatial discretizations are done using SBP operators of 2nd-, 4th and 6th-order accuracy in the interior (see [1]) satisfying

$$D_4 = H^{-1}\left(N + e_l d_{3;l}^\top + d_{1;l} d_{2;l}^\top + e_r d_{3;r}^\top - d_{1;r} d_{2;r}^\top\right),$$

where $H = H^\top > 0$ is a diagonal matrix, $N = N^T \geq 0$, e_l and e_r are the first and last columns of the $m \times m$ identity matrix and $d_{1;l}$, $d_{1;r}$, $d_{2;l}$, $d_{2;r}$, $d_{3;l}$ and $d_{3;r}$ are one-sided finite difference approximations of the first, second and third normal derivatives at the left and right boundary points. Furthermore, the matrix N can be decomposed as

$$\begin{aligned} N = \tilde{N} &+ h\alpha_{II}\left(d_{2;l}d_{2;l}^\top + d_{2;r}d_{2;r}^\top\right) \\ &+ h^3\alpha_{III}\left(d_{3;l}d_{3;l}^\top + d_{3;r}d_{3;r}^\top\right), \end{aligned} \tag{3}$$

where $\tilde{N} = \tilde{N}^\top \geq 0$ and α_{II} and α_{III} are positive constants not dependent on h.

3.1 SBP-P discretization

A consistent and stable semi-discrete approximation of (1) with the interface conditions (2) imposed using the projection method [2] is given by

$$(B \otimes I_m) w_{tt} = -P(A \otimes D_4)Pw, \tag{4}$$

where

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad B = \begin{bmatrix} b^{(1)} & 0 \\ 0 & b^{(2)} \end{bmatrix}, \quad A = \begin{bmatrix} a^{(1)} & 0 \\ 0 & a^{(2)} \end{bmatrix}.$$

The projection operator is given by

$$P = I_{2m} - \bar{H}^{-1} L^\top (L \bar{H}^{-1} L^\top)^{-1} L, \tag{5}$$

where I_{2m} is the $2m \times 2m$ identity matrix and L is given by

$$L = \begin{bmatrix} e_r^\top & -e_l^\top \\ d_{1;r}^\top & d_{1;l}^\top \\ a^{(1)} d_{2;r}^\top & a^{(2)} d_{2;l}^\top \\ a^{(1)} d_{3;r}^\top & a^{(2)} d_{3;l}^\top \end{bmatrix}. \tag{6}$$

3.2 SBP-SAT discretization

A consistent and stable semi-discrete approximation of (1) with the interface conditions (2) imposed using the SAT method [1] is given by

$$\begin{aligned} b^{(1)} u_{tt} &= a^{(1)} D_4 u \\ &\quad - H^{-1}\left(\frac{\tau}{h^3} e_r + \frac{a^{(1)}}{2} d_{3;r}\right)(e_r^\top u - e_l^\top v) \\ &\quad - H^{-1}\left(\frac{\sigma}{h} d_{1;r} - \frac{a^{(1)}}{2} d_{2;r}\right)(d_{1;r}^\top u + d_{1;l}^\top v) \\ &\quad - \frac{1}{2} H^{-1} d_{1;r} (a^{(1)} d_{2;r}^\top u + a^{(2)} d_{2;l}^\top v) \\ &\quad + \frac{1}{2} H^{-1} e_r (a^{(1)} d_{3;r}^\top u + a^{(2)} d_{3;l}^\top v), \\ b^{(2)} v_{tt} &= a^{(2)} D_4 v \\ &\quad - H^{-1}\left(\frac{\tau}{h^3} e_l + \frac{a^{(2)}}{2} d_{3;l}\right)(e_l^\top v - e_r^\top u) \\ &\quad - H^{-1}\left(\frac{\sigma}{h} d_{1;l} + \frac{a^{(2)}}{2} d_{2;l}\right)(d_{1;l}^\top v + d_{1;r}^\top u) \\ &\quad + \frac{1}{2} H^{-1} d_{1;l} (a^{(2)} d_{2;l}^\top v + a^{(1)} d_{2;r}^\top u) \\ &\quad + \frac{1}{2} H^{-1} e_l (a^{(2)} d_{3;l}^\top v + a^{(1)} d_{3;r}^\top u), \end{aligned}$$

where

$$\tau = \frac{1}{4\alpha_{III}}(a^{(1)} + a^{(2)}), \quad \sigma = \frac{1}{4\alpha_{II}}(a^{(1)} + a^{(2)}),$$

and α_{II} and α_{III} are chosen so that $\tilde{N}$ is positive semi-definite.

4 Numerical results

The accuracies of the two methods are evaluated by an analytical solution derived using separation of variables with $a^{(1)} = 1$, $a^{(2)} = 4$ and $b^{(1,2)} = 1$. To isolate the influence of the interface the blocks are coupled at both ends, i.e. at $x = 0$ and $x = \pm 1$, resulting in a periodic problem. The second order ODE systems are integrated using a compact and explicit 4th order accurate finite difference time marching scheme [1]. The time step is chosen such that the spatial error dominates the temporal error.

In Figure 1 the error versus step size is plotted for the 2nd, 4th and 6th order operators with SBP-P and SBP-SAT. The results indicate that the theoretical convergence rates are obtained and that the methods are very similar in terms of accuracy.

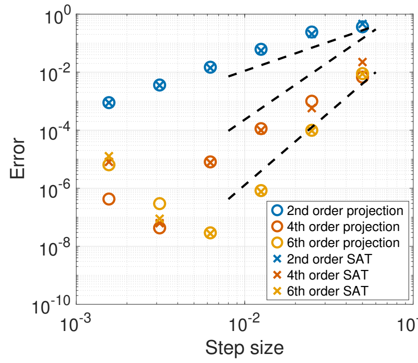

Figure 1: Error versus step size for 2nd, 4th and 6th order SBP-SAT and SBP-P discretizations. The dashed lines indicate the theoretical convergence rates 2, 4 and 5.

5 Conclusions

Two novel and stable methods of imposing interface conditions for the piecewise homogeneous dynamic beam equation are presented. Numerical experiments demonstrates that both methods obtain the expected theoretical convergence rates.

References

[1] K. Mattsson and V. Stiernström, High-fidelity numerical simulation of the dynamic beam equation, *Journal of Computational Physics 286* (2015)

[2] P. Olsson, Summation by parts, projections, and stability. I, *Mathematics of Computation 64 211* (1995) pp. 1035–1065.