Decay Theory and Numerical Analysis for Hybrid Frequency/Time Methods in Long-Time Transient Wave Scattering

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Abstract

Frequency/time hybrid integral-equation methods [1] for transient wave scattering have been demonstrated to provide a highly efficient and accurate, trivially parallelizable means to compute solutions to time-domain scattering problems. We discuss further enhancements for high-fidelity simulation of scattering of long wave-trains via connection to newly-developed results in time-domain scattering theory. The new theory impacts on several long-standing problems in scattering theory, resolving open questions relating to "domain-of-dependence" and decay for problems of wave scattering by bounded, possibly trapping, obstacles. In particular, we present the first rapid decay estimates for connected trapping obstacles and provide norm-bounds on physical surface quantities in terms of measurements over a finite history of time. The theory leads to efficient sum-truncation numerical analysis results for hybrid wave scattering methods that justify an $\mathcal{O}(1)$ asymptotic cost for producing solutions at arbitrarily large times.

Keywords: transient wave propagation, integral equations, decay theory, domain-of-dependence bounds

1 Introduction

Frequency-time hybrid methods [1] for the obstacle scattering problem

$$\frac{\partial^2 u}{\partial t^2}(\mathbf{r},t) - c^2\Delta u(\mathbf{r},t) = 0, \quad \mathbf{r} \in \Omega^c, \tag{1a}$$

$$u(\mathbf{r},0) = \frac{\partial u}{\partial t}(\mathbf{r},0) = 0, \tag{1b}$$

$$u(\mathbf{r},t) = b(\mathbf{r},t) \quad (\mathbf{r},t) \in \Gamma \times [0,T^{inc}], \tag{1c}$$

with Lipschitz boundary $\Gamma = \partial\Omega$, rely on Fourier time transformation of the incident wavefield b in conjunction with certain time-partitioning and windowing techniques which serve to allow solutions at all times t to be computed on the basis of solutions to a fixed set of frequency-domain problems. Briefly, the method uses well-spaced time-window centers $s_k \in [0,T^{inc}]$ and smooth compactly-supported window functions $w_k(t) = w(t - s_k)$ to expand b in a partition of unity representation $b(\mathbf{r},t) = \sum_{k=1}^{K} b_k(\mathbf{r},t)$, where the functions b_k are temporally-localized "wave packets" that also solve (1a) and serve as boundary data for solutions u_k to (1). The solution results by reconstruction via the sum $u(\mathbf{r},t) = \sum_{k=1}^{K} u_k(\mathbf{r},t)$.

We will present efficient means to produce solutions u_k at arbitrary times t at $\mathcal{O}(1)$ cost. Furthermore, it is important to note that since $K = \mathcal{O}(T^{inc})$ the sum-representation of u in principle involves an increasing number of solutions u_k—an issue that we resolve in what follows on the basis of solution decay.

2 3D decay and sum-truncation

In our surface-scattering context, we use the representation formula,

$$u_k(\mathbf{r},t) = (S\psi_k)(\mathbf{r},t), \quad \psi_k = \partial_{\mathbf{n}} u_k^{tot},$$

with S the time-domain single layer potential, from which it follows that if bounds on ψ_k can be established, then the contribution of u_k to the full solution u on certain space-time regions $\mathcal{R} \times \mathcal{T}$ can be neglected with provably small error:

Lemma. *Let $\mathcal{R} \subset \Omega^c$ satisfy $\operatorname{dist}(\mathcal{R},\Gamma) > 0$, and denote by T_0 a given observation time. Then there exists a (known) constant $C(\mathcal{R})$ such that*

$$\sup_{t > T_0 + r_{\max}/c} |u_k(\mathbf{r},t)| \le C \sup_{t > T_0} \|\psi_k(\cdot,t)\|_{L^2(\Gamma)},$$

where $r_{\max} = \max_{\mathbf{r}\in\mathcal{R},\mathbf{r}'\in\Gamma} |\mathbf{r} - \mathbf{r}'|$.

We develop a novel class of time-domain estimates: "domain-of-dependence" estimates that bound ψ_k in terms of its values over a finite time-history of length about $\operatorname{diam}(\Omega)/c$.

We show more, that in fact solutions decay: the decay theorem applies to obstacles satisfying a certain q-growth condition, known to be satisfied by a variety of obstacles.

Theorem ([2]). *Let Γ be the boundary of an obstacle $\Omega \subset \mathbb{R}^3$ satisfying a q-growth condition and assume that the incident wavefield packet b_k is sufficiently smooth. Let I_{T_0} be any time interval (with upper limit T_0) with length exceeding* $\operatorname{diam}(\Omega)/c$ *and laying after the incident packet ceases on Ω. Then for each positive integer n there exists a $C > 0$ independent of b_k and of T_0 such that ψ_k satisfies*

$$\|\psi_k(\cdot,t)\|_{L^2(\Gamma)} \leq C(t-T_0)^{1/2-n}\,\|\psi_k\|_{H^{n(q+1)+1}(I_{T_0};\,L^2(\Gamma))} \quad \textit{for all} \quad t > T_0.$$

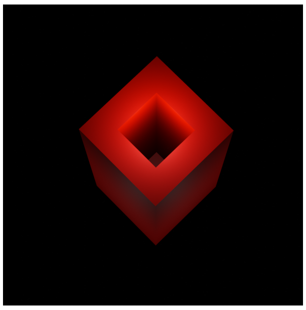

Figure 1: Deep rectangular cavity that satisfies a q-growth condition.

Definition (q-growth condition). *A Lipschitz obstacle Ω satisfies a q-growth condition if there exists a $C > 0$ such that for a non-negative q the frequency-domain combined-field operator A_ω satisfies*

$$\left\|A_\omega^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\omega^q \quad \textit{as} \quad \omega \to \infty.$$

The decay theorem above enables the following numerical analysis result for hybrid frequency/time methods, showing that only an $\mathcal{O}(1)$ number of solutions u_k need to be computed for approximation (within some tolerance $\varepsilon^{\mathrm{tol}}$) of the solution u.

Theorem ([3]). *Let Γ be the boundary of an obstacle that satisfies a q-growth condition. For smooth incident data, a region of space $\mathcal{R}$ of diameter D_r and a time interval $\mathcal{T}$ of length D_t, there exist for every $\varepsilon^{\mathrm{tol}} > 0$ an integer $M(\varepsilon^{\mathrm{tol}}, D_r, D_t)$ and certain integers M_i and M_f satisfying $M_f - M_i = M$ so that for all incident wavefields*

$$\sup_{\substack{t\in\mathcal{T}\\ \mathbf{r}\in\mathcal{R}}}\left|u(\mathbf{r},t) - \sum_{k=M_i}^{M_f-1} u_k(\mathbf{r},t)\right| \leq C(\Gamma, D_r, D_t)\varepsilon^{\mathrm{tol}}.$$

Numerical experiments confirm the guarantees of this theorem for problems with incident wavetrains of many thousands of wavelengths in duration (Figure 2).

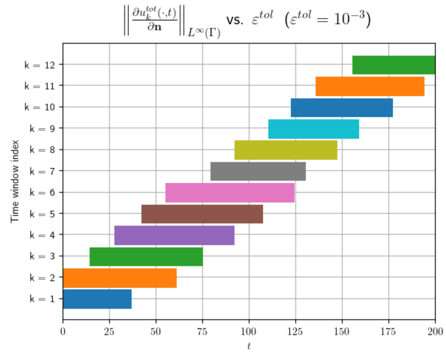

Figure 2: Timeline plot of relevance of individual densities ψ_k to the overall solution u, as measured by being less than a tolerance $\varepsilon^{\mathrm{tol}}$.

3 Classical Scattering Theory

Our primary decay theorem is a result of independent interest in the field of scattering theory, as it is the first decay result not based on the classical Lax-Phillips approach which has not been used to establish wave decay for any of the known trapping obstacles that have resonances lying arbitrarily close to the real axis (i.e. when wave solutions cannot be exponentially decaying). Our techniques are based on real-axis estimates (based on the q-growth condition) and use integration by parts as well as frequency-differentiated boundary integral densities, resulting in the first rapid decay estimates (a) In connected-trapping contexts and (b) In contexts where trapped orbits span the full volume of a physical cube (Figure 1).

References

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