Maximum norm error bounds for the full discretization of non-autonomous wave equations

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Abstract

In this talk, we consider a specific class of non-autonomous wave equations on a smooth, bounded domain and their discretization in space by isoparametric finite elements and in time by the implicit Euler method. Building upon the work of Baker and Dougalis (1980), we prove maximum norm estimates for the semi discretization in space and the full discretization. The key tool is the gain of integrability coming from the inverse of the spatially discretized differential operator. For this, we have to bound differentiated initial errors in the energy norm.

Keywords: error analysis, full discretization, wave equation, maximum norm error bounds, nonconforming space discretization, isoparametric finite elements, a-priori error bounds.

1 Introduction

We consider non-autonomous wave equations of the form

$$\partial_{tt} u(t) = \lambda(t)^{-1} \Delta u(t) + f(t), \tag{1}$$

subject to homogeneous Dirichlet boundary conditions for $t \in [0,T]$ on a domain $\Omega \subset \mathbb{R}^N$, $N = 2,3$, with sufficiently regular boundary Γ. In space, we employ isoparametric finite elements and in time the implicit Euler scheme. Following the approach in [1], we derive maximum norm error bounds for the semi- and full discretization. The main application we have in mind is the quasilinear wave equation

$$\partial_{tt} u(t) = \lambda(u(t))^{-1} \Delta u(t) + f(t, u(t)). \tag{2}$$

In order to guarantee well-posedness of (2), one exploits a pointwise lower bound on $\lambda(u)$ and this property has to be conserved in the discretization. Up to now, this is ensured via inverse estimates that either lead to a CFL conditions or to a restriction on the minimal polynomial degree of the ansatz space. By our linear results, we hope to show that these constraints are only of theoretical nature and can be removed.

2 Discretization in space

We consider the unified error analysis introduced in [3] and reformulate (1) as a first-order system

$$\partial_t y(t) = \Lambda(t)^{-1} \mathrm{A} y(t) + F(t), \tag{3}$$

in the product space $X = H_0^1(\Omega) \times L^2(\Omega)$, with $y = \big(u\,, \partial_t u\big)$ and initial value $y(0) = y^0$, operators

$$\Lambda(t) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \lambda(t) \end{pmatrix}, \qquad \mathrm{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & 0 \end{pmatrix},$$

and $F(t) = \big(0\,, f(t)\big)$. Further, we consider the spatially discretized version on a finite dimensional space X_h

$$\partial_t y_h(t) = \Lambda_h(t)^{-1} \mathrm{A}_h y_h(t) + F_h(t),$$

on the computational domain $\Omega_h \approx \Omega$. In order to relate functions on the two (in general) different domains, we introduce a lift operator $\mathcal{L}_h$ mapping functions on Ω_h to functions on Ω.

For technical reasons, we have to make the following assumption on λ, which ensures the preservation of boundary conditions.

Assumption 1 *There is some $\ell_{max} \geq 4$ such that for $0 \leq \ell \leq \ell_{max}$ and $u \in \mathcal{D}((-\Delta)^{\ell/2})$ it holds*

$$\lambda u, \lambda^{-1} u \in \mathcal{D}\big((-\Delta)^{\ell/2}\big).$$

A sufficient conditions for this assumption is for example given by

$$\nabla \lambda\big|_\Gamma = 0\,.$$

In [2], we obtained the following semidiscrete error bound.

Theorem 2 *Let $\partial\Omega \in C^{k+1,1}$, $\ell_{max} \geq 2$, and let u and λ be sufficiently regular. If the discrete initial value $y_h(0)$ is chosen appropriately, then it holds*

$$\|y(t) - \mathcal{L}_h y_h(t)\|_{L^\infty \times L^\infty} \leq C h^k,$$

where C is independent of h.

3 Discretization in space and time

We denote the time step by $\tau > 0$ and write the implicit Euler method in the form

$$\partial_\tau y_h^n = (\Lambda_h^n)^{-1} \mathrm{A}_h y_h^n + F_h^n, \qquad (4)$$

where we use the discrete derivative

$$\partial_\tau \varphi^n = \frac{1}{\tau}\big(\varphi^n - \varphi^{n-1}\big).$$

For the fully discrete solution (4), we show the following error bound, see [2].

Theorem 3 *Let $\partial\Omega \in C^{k+1,1}$, $\ell_{max} \geq 4$, and let u and λ be sufficiently regular. If the discrete initial value y_h^0 is chosen appropriately, then there is $\tau_0 > 0$ such that for $\tau \leq \tau_0$ and $n \geq 3$ we have the error bound*

$$\|y(t^n) - \mathcal{L}_h y_h^n\|_{L^\infty \times L^\infty} \leq C\tau + Ch^{\min\{k, \ell_{max}-2\}},$$

where C is independent of h and τ, and τ_0 is independent of h.

We note that by a slightly different approach, we prove similar convergence rates for the first approximations y_h^j, $j = 0, 1, 2$.

4 Strategy of the proof

In the continuous case, one can employ Sobolev's embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ to obtain maximum norm bounds on the solution u. However, this is not possible for u_h since the Lagrangian finite elements are *not* H^2-conforming.

Denoting the inverse of the discretized differential operator A_h^{-1}, we can generalize the following result from [1], and still obtain integrability in the discrete case.

Lemma 4 *Let $\partial\Omega \in C^{1,1}$ and $p, q, r \geq 2$ with $0 \leq \frac{1}{r} - \frac{1}{p} < \frac{1}{N}$. Then, it holds for $\xi_h \in X_h$*

$$\big\|\mathrm{A}_h^{-1} \xi_h\big\|_{L^p \times L^q} \leq C \, \|\xi_h\|_{L^q \times L^r} \ .$$

Together with a Sobolev's embedding, a direct consequence of Lemma 4 is the continuity of the map

$$X_h \hookrightarrow L^4(\Omega_h) \times L^2(\Omega_h) \xrightarrow{\mathrm{A}_h^{-3}} L^\infty(\Omega_h) \times L^\infty(\Omega_h)\, .$$

Together with the reformulation of (3) as

$$y_h = \mathrm{A}_h^{-1} \Lambda_h(t) \partial_t y_h - \mathrm{A}_h^{-1} \Lambda_h(t) F_h(t)\, , \qquad (5)$$

we exchange integrability for time derivatives. Denoting the discrete error by $e_h(t)$, the maximum norm is bounded by

$$\|e_h(t)\|_{L^\infty \times L^\infty} \leq C \sum_{j=1}^{3} \big\|\partial_t^j e_h(t)\big\|_{X_h} + Ch^k\, .$$

In the second step, we use energy techniques to bound the time derivatives of the error by further defects and the discrete initial errors

$$\big\|\partial_t^j e_h(0)\big\|_{X_h}, \quad j = 1, 2, 3\, .$$

An appropriately chosen initial value yields the desired order of convergence.

With the implicit Euler scheme given in the form (4), we derive analogously to (5) a representation for the fully discrete scheme. This allows us to mimic the proof of the semi discretization with some calculus for discrete derivatives and obtain the assertion of Theorem 3.

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References

[1] G. A. Baker and V. A. Dougalis, On the L^∞-convergence of Galerkin approximations for second-order hyperbolic equations, *Math. Comp.* **34** (1980), no. 150, pp. 401–424.

[2] B. Dörich, J. Leibold and B. Maier, Maximum norm error bounds for the full discretization of non-autonomous wave equations. CRC 1173 Preprint 2021/47, Karlsruhe Institute of Technology, 2021. URL `https://www.waves.kit.edu/downloads/CRC1173_Preprint_2021-47.pdf`

[3] D. Hipp, M. Hochbruck, and C. Stohrer, Unified error analysis for nonconforming space discretizations of wave-type equations, *IMA J. Numer. Anal.* **39** (2019), no. 3, pp. 1206–1245.