High sensibility imaging of defects in elastic waveguides using near resonance frequencies

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Abstract

This work presents a new multi-frequency inversion method to image shape defects in slowly varying elastic waveguides. Contrary to previous works in this field, we choose to take advantage of the near resonance frequencies of the waveguide, where the elastic problem is known to be ill-conditioned. A phenomenon close to the tunnel effect in quantum mechanics can be observed at these frequencies, and locally resonant modes propagate in the waveguide. These modes are very sensitive to width variations, and measuring their amplitude enables reconstructing the local variations of the waveguide shape with very high sensibility. Given surface wavefield measurements for a range of near resonance frequencies, we provide a stable numerical reconstruction of the width of a slowly varying waveguide and illustrate it on defects like dilation or compression of a waveguide.

Keywords: Waveguide, Inverse problems, Resonances

1 Scientific context

Reconstruction of defects in waveguides is of crucial interest in nondestructive evaluation of structures. This work is based on physical experiments done at Institut Langevin where one tries to reconstruct width defects in thin elastic plates using multi-frequency surface measurements (see [1]). Contrary to usual backscattering methods avoiding resonancies frequencies of the plate, they developed an experimental inversion method using these frequencies to obtain a high sensibility reconstruction of local defects in plates. We here try to provide a mathematical understanding of this inversion method and theoretical results on its stability.

2 Study of the forward problem

We consider an infinite 2D elastic plate with a slowly varying width $2h(x)$ (see and illustration in Figure 2). The wave field $\boldsymbol{u}$ satisfies the elastic equation

$$\begin{cases} -\omega^2 \boldsymbol{u} - \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u})) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{u}).\nu = 0 & \text{on } \partial\Omega, \\ \boldsymbol{u} \text{ is outgoing,} & \end{cases} \qquad (\mathcal{H})$$

where $\boldsymbol{\sigma}(\boldsymbol{u}) = \lambda \operatorname{div}(\boldsymbol{u})\boldsymbol{I} + 2\mu \nabla^s \boldsymbol{u}$ and (λ, μ) are the Lamé coefficients of the plate Ω. Using the formalism $\boldsymbol{X}/\boldsymbol{Y}$ developed in [2], vectors $\boldsymbol{X} := (\boldsymbol{u}_1, (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{e_x})_2)$ and $\boldsymbol{Y} := (-(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{e_y})_1, \boldsymbol{u}_2)$ can be decomposed in Lamb modes $(\boldsymbol{X}_n, \boldsymbol{Y}_n)$:

$$\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{pmatrix}(x,y) = \sum_{n\in\mathbb{N}} \begin{pmatrix} a_n(x)\boldsymbol{X}_n(x,y) \\ b_n(x)\boldsymbol{Y}_n(x,y) \end{pmatrix}. \quad (1)$$

Each Lamb mode is associated to a wavenumber $k_n(x)$, represented in Figure 1.

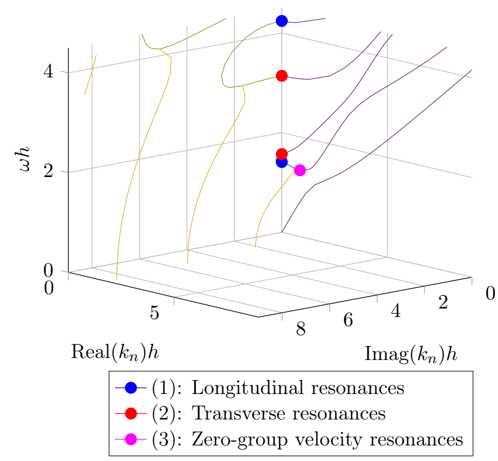

Figure 1: Representation of $k_n(x)$ with respect to $\omega h(x)$. Purple branch: $k_n \in \mathbb{R}$, green branch: $k_n \in i\mathbb{R}$, yellow branch: $k_n \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$.

We are interested in frequencies ω, called resonant frequencies, such that $k_n(x)$ switches from a real number to a complex one when x varies. We distinguish them in three cases, denoted (1), (2), (3) and represented in Figure 1. In each case, we prove that a_n or b_n are close to the solutions of

$$(2),(3): a_n''+k_n^2 a_n = F_1, \; (1),(3): b_n''+k_n^2 b_n = F_2,$$

where F_1 and F_2 depend explicitly on $\boldsymbol{f}$. These are Schrodinger equations, and we show that their solutions can be expressed using Airy functions of the first and second kind, denoted $\mathcal{A}$ and $\mathcal{B}$. It enables an explicit approximation of $\boldsymbol{u}$, denoted $\boldsymbol{u}^{\text{app}}$ (see an example in Figure 2). Adapting the proof of [3], we formally control the error of this approximation:

Theorem 1 *Let $\boldsymbol{f} \in H^2(\Omega_r)$, $h \in \mathcal{C}^2(\mathbb{R})$ such that $\|h'\|_\infty \leq \eta$ and $\|h''\|_\infty \leq \eta^2$. There exists $\eta_0 > 0$ such that if $\eta < \eta_0$, then $(\mathcal{H})$ has a unique solution $\boldsymbol{u} \in H^4_{loc}(\Omega)$ which can be explicitly approached by $\boldsymbol{u}^{app}$. Moreover, there exists a constant $C_1 > 0$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^{app}\|_{L^2(\Omega_r)} \leq \eta C_1 \|\boldsymbol{f}\|_{H^2(\Omega_r)}. \quad (2)$$

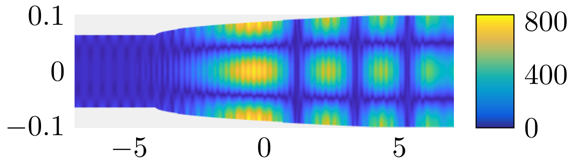

Figure 2: Representation of $|\boldsymbol{u}_1^{\text{app}}|$ at a longitudinal resonance (1) in a slowly increasing waveguide.

3 Imaging of the width of the waveguide

Using the study of the forward problem, we can now provide an approximation of the surface measurements for different frequencies ω. In case (1) (resp. (2), (3)), we see that $d_\omega := \boldsymbol{u}_1$ (resp. $d_\omega := \boldsymbol{u}_2(x)$, $d_\omega := |\boldsymbol{u}_i(x)|$) can be approached by

$$d_\omega(x) \approx \alpha_\omega \mathcal{A}(\beta_\omega(x - x^\star_\omega)), \quad (3)$$

where $\alpha_\omega \in \mathbb{C}$, $\beta_\omega > 0$, and $x^\star_\omega$ is a coordinate such that $k_n(x^\star_\omega)$ is at the junction of two different branches in Figure 1. Since the value of $x^\star_\omega$ is explicitly linked to the value of $h(x^\star_\omega)$ and varies when ω varies, we plan on reconstructing h by finding the value of $x^\star_\omega$ for different frequencies ω using data d_ω. To do so, we numerically minimize the function

$$F : (\alpha_\omega, \beta_\omega, x^\star_\omega) \mapsto \|d_\omega(x) - a_\omega \mathcal{A}(\beta_\omega(x - x^\star_\omega)\|_2.$$

We prove the following result:

Theorem 2 *Under the same assumptions as in Theorem 1, there exists $\eta_1 > 0$ such that if $\eta < \eta_1$, the function F has a unique minimum. If we denote $x^{\star,app}_\omega$ the approximation of $x^\star_\omega$, there exists $C_2 > 0$ independent of ω such that*

$$|x^\star_\omega - x^{\star,app}_\omega| \leq C_2 \eta. \quad (4)$$

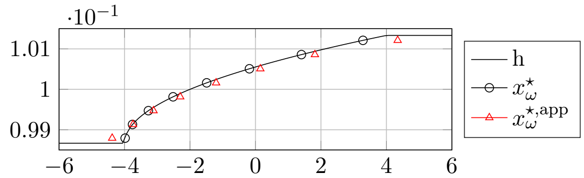

Figure 3: Positions of $x^\star_\omega$ and $x^{\star,\text{app}}_\omega$ for different frequencies ω.

We represent in Figure 3 different points $x^\star_\omega$ and their approximations.

Using the coordinates $x^{\star,\text{app}}_\omega$ and their associated width enables us to provide a stable reconstruction of the width in each case of resonance. We show in Figure 4 two reconstructions using a longitudinal resonance (1).

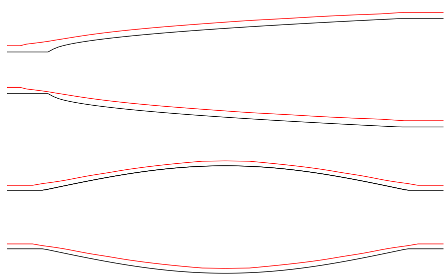

Figure 4: Two reconstructions of waveguide's widths using a longitudinal resonance (1). In black, the original shape of Ω, in red, the reconstruction slightly shifted for comparison purposes.

References

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