Iterative Trefftz Method For Three-dimensional Electromagnetic Waves Simulation.

Sébastien Pernet[1], **Margot Sirdey**[2,*], **Sébastien Tordeux**[3]
[1]ONERA, Toulouse, France
[2]INRIA - EPC MAKUTU, ONERA, E2S UPPA, CNRS
[3]INRIA - EPC MAKUTU, Pau, France
*Email: margot.sirdey@inria.fr

Abstract

The simulation of time-harmonic electromagnetic waves requires a matrix inversion whose cost, especially in three-dimensional cases, increases quickly with the size of the computational domain. This is a major issue regarding memory consumption. Iterative Trefftz methods [3] can overcome this problem when resorting to a GMRES type solver [2]. They can simulate electromagnetic waves in large domains by avoiding the storage of the LU factorization of the matrix.

Keywords: Trefftz method, Electromagnetic waves, Domain Decomposition

1 Maxwell problem

The studied adimensional Maxwell problem is

$$\left\{\begin{array}{rcl} \nabla \times \mathbf{H} &=& ik\mathbf{E} \\ \nabla \times \mathbf{E} &=& -ik\mathbf{H}, \text{ on } \Omega, \end{array}\right. \tag{1}$$

where $\Omega \subset \mathbb{R}^3$, k is the wavenumber, $\mathbf{E}$ and $\mathbf{H}$ are respectively the three-dimensional electric and magnetic fields. We impose an impedance boundary condition, see (3), on the boundary of the domain, denoted by $\partial\Omega$.

The domain Ω is meshed into a set of elements $\mathcal{T}$. It is also decomposed into a set of interior faces $\mathcal{F}_{int}$ and a set of exterior faces $\mathcal{F}_{ext}$. We define in each element $T \in \mathcal{T}$ the electromagnetic field $\mathbb{E}^T := (\mathbf{E}^T, \mathbf{H}^T)$ whose tangential electric and magnetic traces are defined as $\gamma_t\mathbf{E}^T = (\mathbf{n}_T \times \mathbf{E}^T) \times \mathbf{n}_T$ and $\gamma_\times^T\mathbf{H}^T = \mathbf{n}_T \times \mathbf{H}^T$, with $\mathbf{n}_T$ the outward unit normal to T.

2 Ultra-Weak Variational Formulation

Cessenat and Després developed in [1] the Ultra-Weak Variational Formulation (UWVF) method based on incoming and outgoing trace operators respectively defined as following

$$\begin{array}{rcl} \gamma_{in}^T\mathbb{E}^T &=& \gamma_t\mathbf{E}^T + \gamma_\times^T\mathbf{H}^T, \\ \gamma_{out}^T\mathbb{E}^T &=& \gamma_t\mathbf{E}^T - \gamma_\times^T\mathbf{H}^T. \end{array} \tag{2}$$

The boundary condition can be written as

$$\gamma_{in}^T\mathbb{E}^T = R\gamma_{out}^T\mathbb{E}^T + \mathbf{f} \quad \text{on } \partial\Omega \cap \partial T, \tag{3}$$

with $\mathbf{f} \in \mathcal{L}^2(\partial\Omega)$ a tangential field and R the reflexion coefficient.

A Trefftz method is defined by a Trefftz space $\mathbb{X}_T$ and an ultra-weak energy conservation property. The Trefftz space is made of local solutions of (1) which are taken as analytical [3] plane waves [4] in our work. The energy property is given by

Theorem 1 *For all $\mathbb{E} \in \mathbb{X}_T$ and $\mathbb{E}' \in \mathbb{X}_T$*

$$\sum_{T\in\mathcal{T}} \int_{\partial T} \gamma_{out}^T\mathbb{E}^T \cdot \overline{\gamma_{out}^T\mathbb{E}'^T} = \sum_{T\in\mathcal{T}} \int_{\partial T} \gamma_{in}^T\mathbb{E}^T \cdot \overline{\gamma_{in}^T\mathbb{E}'^T}.$$

Hence, the UWVF is *Find $\mathbb{E} \in \mathbb{X}_T$ such that*

$$a(\mathbb{E}, \mathbb{E}') = \ell(\mathbb{E}') \quad \forall \mathbb{E}' \in \mathbb{X}_T, \tag{4}$$

where

$$a(\mathbb{E}, \mathbb{E}') = \sum_{T\in\mathcal{T}} \left(a^T(\mathbb{E}, \mathbb{E}') - b^T(\mathbb{E}, \mathbb{E}')\right), \tag{5a}$$

$$a^T(\mathbb{E}, \mathbb{E}') = \int_{\partial T} \gamma_{out}^T\mathbb{E}^T \cdot \overline{\gamma_{out}^T\mathbb{E}'^T}, \tag{5b}$$

$$b^T(\mathbb{E}, \mathbb{E}') = \int_{\partial T} \widetilde{\gamma_{in}^T\mathbb{E}^T} \cdot \overline{\gamma_{in}^T\mathbb{E}'^T}, \tag{5c}$$

$$\ell(\mathbb{E}') = \sum_{T\in\mathcal{T}} \int_{\partial T\cap\partial\Omega} \mathbf{f} \cdot \overline{\gamma_{in}^T\mathbb{E}'^T}. \tag{5d}$$

The numerical flux $\widetilde{\gamma_{in}^T\mathbb{E}^T}$ ensures the consistency of the method. It imposes the continuity on interior faces and the impedance boundary condition (3) on exterior faces

$$\widetilde{\gamma_{in}^T\mathbb{E}^T} = \left\{\begin{array}{ll} \gamma_{out}^K\mathbb{E}^K & \text{on } \partial T \cap \partial K \in \mathcal{F}_{int}, \\ R\gamma_{out}^T\mathbb{E}^T & \text{on } \partial\Omega \cap \partial T \in \mathcal{F}_{ext}. \end{array}\right.$$

3 Iterative Trefftz method

Problem (4) is written into the matricial form

$$(M + N)X = F, \tag{6}$$

where M is the matrix associated to (5b), N is the matrix associated to (5c) and F is the vector associated to (5d). Classically, (6) is solved by the Cessenat and Després fixed point algorithm

$$X^{n+1} = M^{-1}NX^n + F, \quad X^0 = 0, \qquad (7)$$

which converges since $M^{-1}N$ is contractant, see Fig. 1 and [1]. However, eigenvalues close to the unit circle tend to slow down the convergence. A GMRES solver [2] performs better but still has difficulties to deal with small eigenvalues (in red in Fig. 1). Consequently, a new global preconditioner has been developed to accelerate the convergence of the iterative process, see Fig. 1.

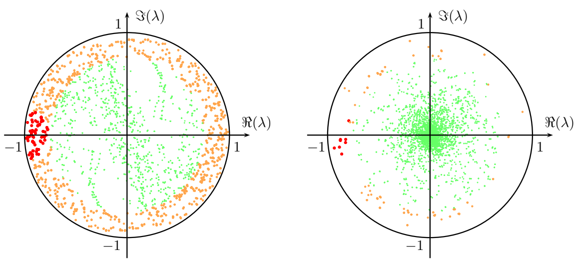

Figure 1: Cessenat and Després preconditioner (left) and global preconditioner (right).

Another major issue of UWVF method is related to rounding errors. The matrix $M^{-1}N$ is ill-conditioned since the plane waves Galerkin basis functions are numerically linearly-dependant. We present a new basis reduction inspired from [4] improving the local conditioning. Indeed, when considering 196 plane waves for a domain with size λ and a mesh of size $h = \frac{\lambda}{4}$, the sparse solver MUMPS can even not invert the matrix without reduction due to a poor conditioning. When lunching the GMRES solver, the improvement when using reduction is clear, see Fig. 2. It leads to a robust iterative method and to a computational cost reduction.

4 Numerical results

In the presentation, we will focus on the memory cost, used to compute the numerical solution of (1), for different numerical methods when the domain size increases with respect to the wavelength λ, see Fig. 3. Namely, we will compare a Low and High Order Nédélec Finite Elements method, a LU Trefftz method and a GMRES Trefftz method using Cessenat and Després decomposition. These numerical results clearly show the potential of such iterative method.

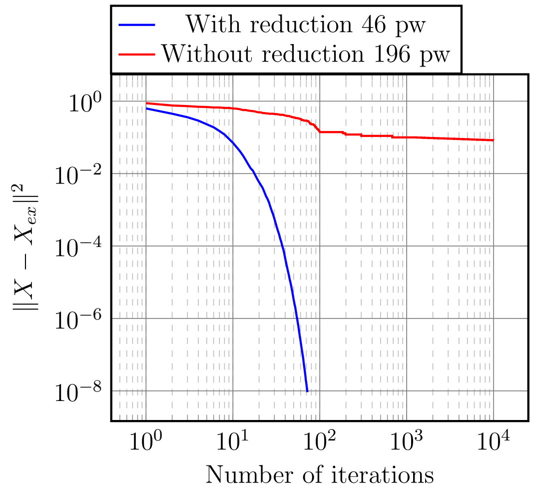

Figure 2: Conditioning gain when using reduction in GMRES solver for $\lambda = 1$ and $h = 0.25$.

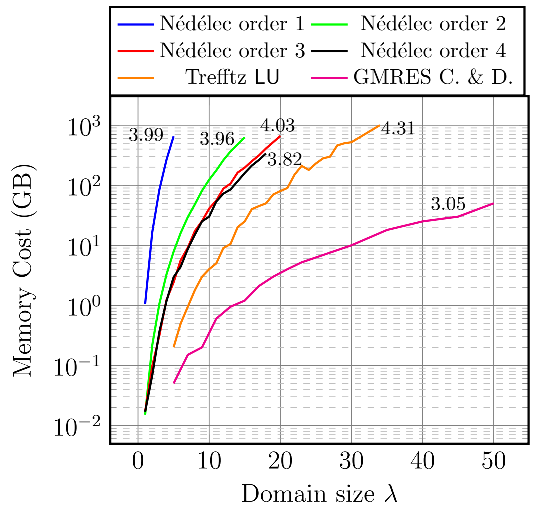

Figure 3: Memory cost for solving (1) for different methods with 1% accuracy error.

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