Acoustoelasticity of soft viscoelastic solids and phononic crystals

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Abstract

The effective dynamic properties of specific periodic structures involving rubber-like materials can be adjusted by pre-strain, thus facilitating the design of custom acoustic filters. While nonlinear viscoelastic behaviour is one of the main features of soft solids, it has been rarely incorporated in the study of such phononic media. Here, we study the dynamic response of nonlinear viscoelastic solids within a 'small-on-large' acoustoelasticity framework. Incompressible soft solids whose behaviour is described by the Fung–Simo quasi-linear viscoelasticy theory (QLV) are considered. Generalised Maxwell-type wave dispersion is obtained, where the coefficients depend on the large static pre-strain. The acoustoelasticity theory is then applied to phononic crystals involving a lattice of hollow cylinders. Results highlight the effect of viscoelastic dissipation on the first band gap. In particular, we show that dissipation increases the band gap width.

Keywords: viscoelasticity, phononic crystal, finite strain

1 Introduction

Rubber-like solids are very soft. Due to their high strength, they can support very large elastic deformations. For specific phononic crystals, the effective wave filtering properties can be adjusted by imposing a large static pre-deformation. Known as the *acoustoelastic effect*, the influence of static pre-deformation on wave dispersion results from nonlinear constitutive behaviour. Such a system is presented in Barnwell et al. [1] within the framework of incompressible finite elasticity, where vertical pre-stretch is applied to a lattice of hollow cylinders embedded in a soft matrix (Fig. 1).

In addition to their nonlinear elastic response, soft solids can exhibit large hysteresis loops in loading-unloading experiments, as well as creep and relaxation phenomena. These

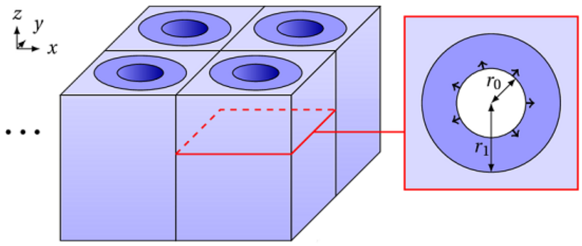

Figure 1: Phononic crystal [1]. Pre-stretched hollow cylinders embedded in a solid matrix.

observations suggest that dedicated large strain *viscoelasticity* theories should be used to study the deformation and motion of real elastomers, as well as related soft phononic crystals.

In the present study [2], we consider soft solids governed by the Fung–Simo quasi-linear viscoelasticity theory (QLV). We derive the incremental equations using stress-like memory variables governed by linear evolution equations. Then, we apply the acoustoelasticity theory to the periodic structure of Fig. 1 by means of Bloch wave analysis and perturbation theory. Results are relevant to practical applications of soft viscoelastic solids subject to static pre-stress.

2 Governing equations

Let us introduce the deformation gradient tensor $\boldsymbol{F} = \partial \boldsymbol{x}/\partial \boldsymbol{X}$ describing a finite motion from the initial position $\boldsymbol{X}$ in the reference configuration to the current position $\boldsymbol{x}$. The incompressible QLV theory involves stress-like memory variables $\boldsymbol{S}^{\mathrm{v}}_\ell$ arising naturally in the expression of the second Piola–Kirchhoff stress [2]

$$\boldsymbol{S} = -p\boldsymbol{C}^{-1} + \mathrm{Dev}(\boldsymbol{S}^{\mathrm{e}}) - \sum_{\ell=1}^{N} \boldsymbol{S}^{\mathrm{v}}_\ell . \qquad (1)$$

Here, $p\boldsymbol{C}^{-1}$ is a hydrostatic stress due to incompressibility, $\mathrm{Dev}(\bullet) = (\bullet) - \frac{1}{3}\mathrm{tr}(\bullet\,\boldsymbol{C})\boldsymbol{C}^{-1}$ is the Lagrangian deviatoric operator, and $\boldsymbol{C} = \boldsymbol{F}^\top\boldsymbol{F}$ is the right Cauchy–Green strain tensor. The elastic response is assumed of Mooney–Rivlin

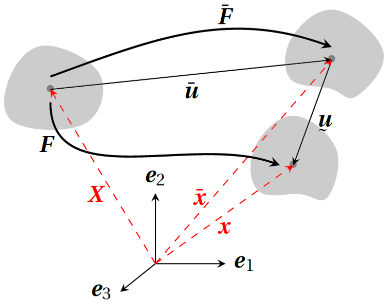

Figure 2: Acoustoelasticity. Incremental motion superimposed on a large static deformation.

type, i.e.

$$\boldsymbol{S}^{\mathrm{e}} = (2\mathfrak{C}_1 + 2\mathfrak{C}_2 \operatorname{tr} \boldsymbol{C})\, \boldsymbol{I} - 2\mathfrak{C}_2 \boldsymbol{C}. \tag{2}$$

The memory variables are governed by linear evolution equations of the form

$$\tau_\ell \dot{\boldsymbol{S}}_\ell^{\mathrm{v}} = g_\ell \operatorname{Dev}(\boldsymbol{S}^{\mathrm{e}}) - \boldsymbol{S}_\ell^{\mathrm{v}}, \tag{3}$$

where the overdot denotes the material time derivative. Finally, the motion is governed by the equation of motion $\rho\ddot{\boldsymbol{u}} = \operatorname{Div}(\boldsymbol{F}\boldsymbol{S})$ where $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ is the displacement field, ρ is the mass density, and the divergence is computed with respect to the Lagrangian coordinate $\boldsymbol{X}$. Typical values of the parameters $\mathfrak{C}_1$, $\mathfrak{C}_2$, τ_ℓ, g_ℓ, ρ for rubber are given in Ref. [2] where $N = 1$ relaxation mechanism (3) is considered.

3 Incremental motion of a pre-stressed phononic crystal

Let us consider an infinitesimal vertical displacement $\underset{\sim}{\boldsymbol{u}}$ superimposed on a large static pre-deformation $\bar{\boldsymbol{F}}$ (Fig. 2), here a vertical pre-stretch of the cylinders. The cylindrical coordinate system r, θ of a cylinder is used. The effective radial and angular shear moduli take the form [2]

$$\mu_\alpha = (1 - g_1)[\bar{T}_{\mathrm{d}}^{\mathrm{e}}]_{\alpha\alpha} + \left(1 - \frac{g_1}{1 + \mathrm{i}\omega\tau_1}\right) \bar{\mu}_\alpha^{\mathrm{v}}\,, \tag{4}$$

where the real coefficients $[\bar{T}_{\mathrm{d}}^{\mathrm{e}}]_{\alpha\alpha}$, $\bar{\mu}_\alpha^{\mathrm{v}}$ with $\alpha \in \{r, \theta\}$ depend on the static pre-stretch, ω is the angular frequency, and i is the imaginary unit — here, we have set $N = 1$ in Eq. (1). Thus, the dispersion of antiplane shear waves is governed by a generalised Maxwell relation whose coefficients depend on $\bar{\boldsymbol{F}}$.

These expressions are then used to perform Bloch wave analysis. The dispersion of plane shear waves propagating in the periodic structures is governed by an eigenvalue problem of the form

$$(\mathbf{K} - \omega^2\mathbf{M})\, \mathbf{w} = \mathbf{0}\,, \tag{5}$$

with complex-valued and ω-dependent matrices $\mathbf{K}$, $\mathbf{M}$ in the viscoelastic[1] case $g_1 \neq 0$.

In practice, the angular frequency ω in Eq. (5) is computed by implementing a perturbation method with respect to the small parameter g_1. For sake of simplicity, we assume that the viscoelastic parameters g_1, τ_1 are identical in the cylinders and in the matrix, contrary to the elastic parameters $\mathfrak{C}_i$, ρ. The dispersion diagrams include a *band gap*, i.e. a frequency range for which waves cannot propagate.

Results show that the band gap width increases monotonously with g_1 for constant τ_1. Moreover, we demonstrate that the evolution of the band gap width with respect to τ_1 at constant g_1 is not monotonous.

4 Concluding remarks

Viscoelastic behaviour does not modify dramatically the dynamic response of the phononic crystal. Nevertheless, dissipation definitively influences the band gap width, a property to keep in mind for practical applications.

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[1]These matrices are real-valued and constant in the elastic case $g_1 = 0$ [1].