# Scattering of an Ostrovsky Wave Packet in a Delaminated Waveguide

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**Abstract**

Layered structures are highly dependant on the material and bonding between each solid waveguide. In this work we will analyse a two layered waveguide with a delamination in the centre and soft (imperfect) bonding either side of the centre. The lower layer of the waveguide is assumed to be significantly denser than the upper layer, leading to a system of Boussinesq-Klein-Gordon (BKG) and Boussinesq equations. Direct numerical modelling is difficult and so we will use a semi-analytical approach consisting of several matched asymptotic multiple-scale expansions, which leads to Ostrovsky equations in soft bonded regions and Korteweg-de Vries equations in the delaminated region. We will also discuss how the dispersion relation is used to determine the wave speed and hence classify the length of the delamination, in addition to changes in the amplitude of the wave packet. These results can provide a tool to control the integrity of layered structures.

*Keywords:* Boussinesq-Klein-Gordon equation; multiple-scales expansions; Ostrovsky wave packet.

## 1 Introduction

The discovery of solitons as localised stable structures [1] in combination with theoretical results of the propagation of long longitudinal bulk solitary waves in elastic waveguides [2] has provided a gateway to explore how the quality of bonding effects the integrity of a layered structure and recent experiment have confirmed the existence of solitons in layered waveguides [3].

In this work we will investigate a two layered bar with a initial delaminated region followed by delamination in the centre of soft bonding, where the material between the layers is distinctly different. The strain waves are described by Boussinesq-Klein-Gordon (BKG) equations in bonded regions and Boussinesq equations in the delaminated regions. A weakly nonlinear solution will be used to model the propagating waves in each section. Then we will compare a semi analytical solution, that makes use of the weakly-nonlinear solution, and a direct numerical solution.

Long waves propagate within these waveguides and the initial solitary wave undergoes a nonlinear steepening process, producing a wave packet consisting of a long wave envelope, through which shorter, faster solitary-like waves propagate [4]. This wave packet emerges as a solution of the Ostrovsky equation

$$(u_t + \alpha u u_x + \beta u_{xxx})_x = \gamma u. \qquad (1)$$

The Ostrovsky equation was originally applied in the context of shallow internal and surface water waves as the effect of the Earth's rotation was considered [5]. The envelope measures the amplitude, and the ratio of the carrier wavelength. The scattering of an Ostrovsky wave packet is present in the "soft" bonded region and generated from a solitary wave and the evolution of wave packets generated by an initial pulse has been thoroughly examined [6].

## 2 Weakly Nonlinear Solution

Consider the scattering of a long longitudinal strain solitary wave in a two layered bar with delamination in the centre as show in Figure 1.

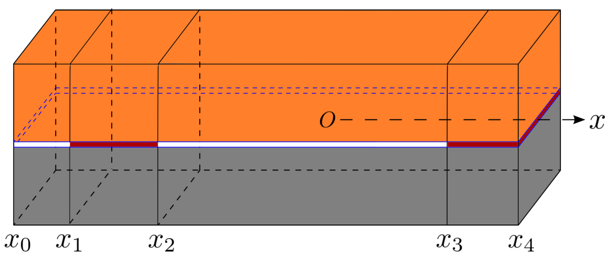


Figure 1: Schematic of the bi-layer with initial small delaminated region followed by a delaminated region between two soft bonded regions.

The strain of the nonlinear wave propagating within the upper waveguide layer can be modelled by system of Boussinesq equations

$$f^{(i)}_{tt} - f^{(i)}_{xx} = 2\varepsilon \left(-3\left(f^{(i)^2}\right)_{xx} + f^{(i)}_{ttxx}\right), \qquad (2)$$

where $i = 1, 3$, and BKG equations

$$f_{tt}^{(j)} - f_{xx}^{(j)} = 2\varepsilon \left( -3 \left( {f^{(j)}}^2 \right)_{xx} + f_{ttxx}^{(j)} - \gamma f^{(j)} \right), \tag{3}$$

where $j = 2, 4$, with continuity conditions on the interface between the sections.

Substituting multiple-scale expansions into (3) and utilising space averaging yields the following Ostrovsky equations

$$(T_X^{(j)} - 6T^{(j)} T_\xi^{(j)} + T_{\xi\xi\xi}^{(j)})_\xi = \gamma T^{(j)}, \tag{4}$$

$$(R_X^{(j)} - 6R^{(j)} R_\eta^{(j)} + R_{\eta\eta\eta}^{(j)})_\eta = \gamma R^{(j)}, \tag{5}$$

where $T_X^{(j)}$ and $R_X^{(j)}$ as the leading order transmitted and reflected waves respectively.

Similarly for (2) we get

$$T_X^{(i)} - 6T^{(i)} T_\xi^{(i)} + T_{\xi\xi\xi}^{(i)} = 0, \tag{6}$$

$$R_X^{(i)} - 6R^{(i)} R_\eta^{(i)} + R_{\eta\eta\eta}^{(i)} = 0, \tag{7}$$

which are KdV equations, describing the leading order transmitted waves $T_X^{(i)}$ and reflected waves $R_X^{(i)}$.

## 3 Numerical Methods and Results

Numerically solving (2) - (3) directly using the finite difference method presented in [7] and comparing to the semi-analytical method gives the results outlined in Figure 2

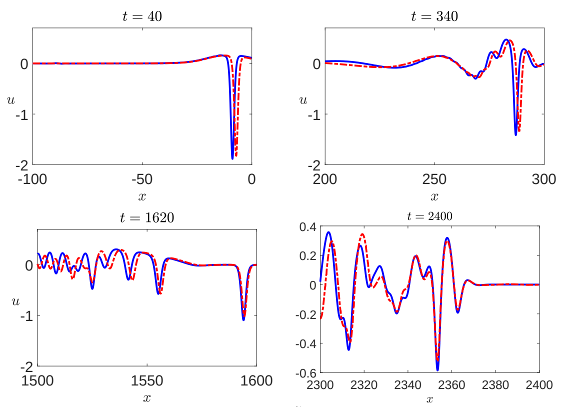


Figure 2: Longitudinal strain waves in a two layered waveguide with finite delamination for direct numerical simulations (blue, solid line) and semi-analytical solution (red, dashed line).

In Figure 2 initially, when the wave is in the delaminated region, the wave behaviour is soliton-like. Once the wave enters the first soft bonded region, the wave then evolves in an Ostrovsky wave. Once the wave re-enters the delaminated region for a sufficient duration, multiple solitons are generated behind a leading soliton. Then, when the wave re-enters a soft bonded region, we see at $t = 2400$ the leading wave evolves into an Ostrovsky wave packet.

The wave speed can be calculated from the simulation in Figure 2 as the waveguide length and time of propagation is known. The numerical wave speed can be compared to the wave speed obtained using the dispersion relation, which as suggested in [6] should output similar results for the speed. We see they are in good agreement.

Also, the delamination length can be varied to provide a further insight into the structure's integrity by analysing the amplitude change and phase shift of the leading waves. Numerical results of this will be provided at the talk.

## References

[1] N. J. Zabusky and M. D. Kruskal, Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States, *Phys. Rev. Lett.* **15** 240–243 (1965).

[2] A. M. Samsonov, Strain Solitons in Solids and How to Construct Them, *Chapman and Hall/CRC, Boca Raton, FL* (2001).

[3] G. V. Dreiden, Bulk strain solitary waves in bonded layered polymeric bars with delamination, *J. Appl. Phys.* **112(6)** (2012).

[4] R. H. J. Grimshaw and Y. A. Stephanyants, Emergence of Envelope Solitary Waves from Initial Localized Pulses within the Ostrovsky Equation, *Radiophys. Quantum. El* **63(1)** 21–28 (2020).

[5] L. A. Ostrovsky, Nonlinear internal waves in a rotating ocean, *Oceanology* **18** 119–125 (1978).

[6] R. Grimshaw and K. Helfrich, Long-time solutions of the Ostrovsky equation *Stud. Appl. Math.* **121** 71-88 (2008).

[7] M.R. Tranter, Solitary wave propagation in elastic bars with multiple sections and layers, *Wave Motion* **86** 21–31 (2019).