Boundary stabilization of critical nonlinear JMGT equation with undissipated Neumann boundary

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Abstract

Boundary feedback stabilization Jordan–Moore–Gibson–Thompson (JMGT) equation in the nonlinear and critical case is considered. The boundary feedback is supported only on a portion of the boundary, while its remaining is left free (available to control actions) and fail to satisfy Lopatinski condition (unlike Dirichlet boundary conditions) making the analysis of uniform stabilization from the boundary to become very subtle and to require careful geometric considerations.

Keywords: boundary feedback stabilization, nonlinear acoustics , JMGT–equation

1 Introduction

The JMGT equation is a third–order (in time) semilinear PDE, a established model for nonlinear acoustics (NLA) which has been recently widely studied [1, 2, 3, 4, 5, 6, 9]. Here, *critical* refers to the usual case where media–damping effects are non–existent or difficult to measure.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with smooth boundary $\Gamma := \partial\Omega$ and consider the semilinear JMGT–equation

$$\begin{aligned} \tau u_{ttt} + (\alpha - 2ku)u_{tt} \\ -c^2\Delta u - (\delta + \tau c^2)\Delta u_t = 2ku_t^2, \end{aligned} \tag{1}$$

where $c, \delta, k > 0$ are constants representing the speed and diffusivity of sound and a nonlinearity parameter, respectively, while the function $\alpha : \overline{\Omega} \to \mathbb{R}^+$ accounts for natural frictional. The parameter $\tau > 0$ accounts for thermal time relaxation.

For the analysis of long–time dynamics the function

$$\gamma : \overline{\Omega} \to \mathbb{R}, \qquad \gamma(x) \equiv \alpha(x) - \frac{\tau c^2}{b} \tag{2}$$

plays a central role. In fact, for zero Neumann or Dirichlet data, if $\gamma(x) \geqslant \gamma_0 > 0$ a.e. in Ω both linear $(k = 0)$ and nonlinear dynamics are uniform exponentially stable [8]. If $\gamma < 0$, chaotic solutions might appear [7] and if $\gamma \equiv 0$ then the energy is conserved. What mechanisms could ensure stability when $\gamma(x) \geqslant 0$ (critical).

In this work we study the stabilizability property of the following boundary conditions

$$\begin{aligned} \lambda\partial_\nu u + \kappa_0(x)u = 0 \text{ on } \Sigma_0 \\ \partial_\nu u + \kappa_1(x)u_t = 0 \text{ on } \Sigma_1 \end{aligned} \tag{3}$$

with $\Gamma_0, \Gamma_1 \subset \Gamma$ relatively open, $\Gamma_0 \neq \emptyset$, $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma, \Gamma_0 \cap \Gamma_1 = \emptyset$, $\lambda > 0$, $\kappa_0 \in L^\infty(\Gamma_0)$ and $\kappa_1 \in L^\infty(\Gamma_1)$, $\kappa_1(x) \geq \kappa_1 > 0, \kappa_0 > 0$ a.e.

2 Functional Analytic Setting

We consider the system comprised of (1), boundary conditions (3) and initial conditions

$$u(0, \cdot) = u_0, u_t(0, \cdot) = u_1, u_{tt}(0, \cdot) = u_2 \tag{4}$$

with regularity to be specified in what follows.

Let A be extension (by duality) of the negative Laplacian with domain

$$\mathcal{D}(A) = \begin{aligned} \{\xi \in H^2(\Omega);\ \partial_\nu\xi|_{\Gamma_1} = 0, \\ [\partial_\nu\xi + \kappa_0\xi]_{\Gamma_0}] = 0\} \end{aligned} \tag{5}$$

and let phase space $\mathbb{H}$ given by

$$\mathbb{H} := \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega) \tag{6}$$

with

$$\|u\|^2_{\mathcal{D}(A^{1/2})} := \|\nabla u\|_2^2 + \int_{\Gamma_0} \kappa_0|u|^2 d\Gamma_0 \sim \|u\|^2_{H^1(\Omega)}$$

The u–problem $(k = 1/2)$ can be written as

$$\begin{aligned} \tau u_{ttt} + \alpha u_{tt} + c^2Au \\ +bAu_t + c^2AN(\kappa_1 N^*Au_t) \\ +bAN(\kappa_1 N^*Au_{tt}) = u_t^2 + uu_{tt} \end{aligned} \tag{7}$$

which we transform into the first order Cauchy–problem

$$\begin{cases} \Phi_t = \mathcal{A}\Phi + \mathcal{F}(\Phi) \\ \Phi(0) = \Phi_0 = (u_0, u_1, u_2)^\top, \end{cases} \tag{8}$$

in the variable $\Phi = (u, u_t, u_{tt})^\top$ with $\mathcal{F}(\Phi)^\top \equiv (0, 0, \tau^{-1}(u_t^2 + uu_{tt}))$ and $\mathcal{A}$ with action (on $\vec{\xi} = (\xi_1, \xi_2, \xi_3)^\top$) and domain respectively

$$\mathcal{A}\vec{\xi} := \begin{pmatrix} \xi_2, \xi_3, -\frac{c^2}{\tau}A\xi_1 \\ -\frac{c^2}{\tau}AN(\kappa_1 N^*A)\xi_2 - \frac{b}{\tau}A\xi_2 \\ -\frac{b}{\tau}AN(\kappa_1 N^*A)\xi_3 - \frac{\alpha}{\tau}\xi_3 \end{pmatrix}^\top \tag{9}$$

$$\mathcal{D}(\mathcal{A}) = \begin{cases} \vec{\xi} \in \left[H^2(\Omega)\right]^2 \times H^1(\Omega); \\ [\partial_\nu \xi_1 + \kappa_0 \xi_1]_{\Gamma_0} = [\partial_\nu \xi_2 + \kappa_0 \xi_2]_{\Gamma_0} = 0 \\ [\partial_\nu \xi_1 + \kappa_1 \xi_2]_{\Gamma_1} = [\partial_\nu \xi_2 + \kappa_1 \xi_3]_{\Gamma_1} = 0 \end{cases} \quad (10)$$

In order to treat the nonlinear problem we consider a second phase space and its norm

$$\mathbb{H}_1 = \begin{cases} \vec{\xi} \in \mathbb{H};\, \Delta\xi_1 \in L^2(\Omega);\, [\lambda \partial_\nu \xi_1 + \kappa_0 \xi_1]_{\Gamma_0} = 0; \\ [\partial_\nu \xi_1 + \kappa_1 \xi_2]_{\Gamma_1} = 0 \end{cases} \quad (11)$$

$$\|\vec{\xi}\|^2_{\mathbb{H}_1} = \|\vec{\xi}\|^2_{\mathbb{H}} + \|\Delta\xi_1\|^2_2 + \|\partial_\nu \xi_1\|^2_{H^{1/2}(\Gamma)}$$

3 Main Results

Assumption 1 *The boundary Γ_0 is star–shaped and convex. In addition, there exists a convex level set function which defines Γ_0. See* [3, 10].

Theorem 1 (Two level uniform stability) *Let Assumption 1 be in force and $\gamma(x) \geqslant 0$. Then the operator $\mathcal{A}$ generates uniformly exponentially stable semigroups on both $\mathbb{H}$ and $\mathbb{H}_1$.*

Given $T > 0$, we say that $\Phi(t)$ is a **mild solution** for the system (1), (3) and (4) provided $\Phi \in C([0,T], \mathbb{H}_1)$ and Φ is given by the variation of parameter (VofP) formula correspoding to the solution of (8) with underlying semigroup being the one generated by $\mathcal{A}$ in $\mathbb{H}_1$.

Define $\mathbb{H}^\rho := \{\Phi \in \mathbb{H}_1; \|\Phi\|_{\mathbb{H}} < \rho\}$ $(\rho > 0)$.

Theorem 2 (Global Solutions) *Let Assumption 1 be in force. Then, there exists $\rho > 0$ sufficiently small* (*depending on the parametrs in the equation*) *such that, given any $\Phi_0 \in \mathbb{H}^\rho$ the VofP formula defines a continuous $\mathbb{H}_1$–valued mild solution for the system* (1), (3) *and* (4). *Moreover, for such $\rho > 0$, there exists $R = R(\|\Phi_0\|_{\mathbb{H}_1})$ such that all trajectories starting in $B_{\mathbb{H}^\rho}(0, R)$remain in $B_{\mathbb{H}^\rho}(0, R_1)$ for all $t \geqslant 0$ and R_1, R are such that $R_1 > R$.*

Theorem 3 (Nonlinear Uniform Stability) *Let Assumption 1 be in force and assume $\gamma \in L^\infty(\Omega)$ and $\gamma(x) \geqslant 0$. Then, there exists $\rho > 0$ sufficiently small and $M(\rho), \omega > 0$ such that if $\Phi_0 \in \mathbb{H}^\rho$ then*

$$\|\Phi(t)\|_{\mathbb{H}_1} \leqslant M(\rho) e^{-\omega t} \|\Phi_0\|_{\mathbb{H}_1}, \qquad t \geqslant 0 \quad (12)$$

where Φ is the mild solution given by Theorem 2.

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