Stable approximation of Helmholtz solutions with evanescent plane waves

Emile Parolin[1,*], **Daan Huybrechs**[2], **Andrea Moiola**[1]

[1]Università degli studi di Pavia, Pavia, Italy

[2]KU Leuven, Leuven, Belgium

[*]Email: emile.parolin@unipv.it

Abstract

Helmholtz solutions are known to be well approximated by a suitable finite superposition of (propagative) plane waves, leading to successful Trefftz methods. Yet, when too many plane waves are used, the computation of the approximation is known to be numerically unstable. This comes from the presence of (exponentially) large coefficients in the expansion.

We show that any Helmholtz solution on a disk can be exactly represented by a continuous superposition of plane waves, provided that evanescent ones (with a complex-valued propagation direction) are included. This generalizes the standard Herglotz representation. Besides, the operator mapping the Helmholtz solution to its extended Herglotz density is invertible and continuous. While the result holds at the continuous level, such a property paves the way for accurate discrete approximation expansions that are moreover stable.

Keywords: Helmholtz equation, Evanescent plane waves, Stable approximation

1 Introduction

We consider the numerical approximation of solutions u of the homogeneous Helmholtz equation $-\Delta u - \kappa^2 u = 0$ with wavenumber $\kappa > 0$. As a model problem, the domain of propagation is the unit disk B_1.

A well-studied way to represent Helmholtz solutions is to approximate them with linear combinations of *propagative plane waves* (PPW) $\mathbf{x} \mapsto e^{\imath\kappa\mathbf{d}\cdot\mathbf{x}}$, for different propagation directions $\mathbf{d}(\varphi) := (\cos\varphi, \sin\varphi) \in \mathbb{R}^2$ parametrized by the angle $\varphi \in [0, 2\pi)$. The main reason is that PPW offer better accuracy for less degrees of freedom in comparison to polynomial spaces and their simple exponential expressions allow very cheap implementations, in comparison to other particular solutions [3]. In 2D, isotropic approximations are obtained by using equispaced angles: for some $M \in \mathbb{N}$, $\varphi_m := \frac{2\pi m}{M}$, $1 \leq m \leq M$. Explicit *hp*-estimates in suitable Sobolev seminorms are available for general domains ensuring exponential convergence (with respect to the number of PPW used) of the approximation of homogeneous Helmholtz solutions [3, §3.2]. Therefore, at least in principle, PPW are well-suited for Trefftz approximations.

2 Propagative plane waves are unstable

The computation of PPW expansions is known to be numerically unstable when increasing the size of the approximation space [3, §4.3] and this is perhaps the main reason that prevented a widespread use of plane-wave based Trefftz schemes. The issue is often understood as an effect of the ill-conditioning of the underlying linear system, which necessarily arises from the almost linear dependence of propagative plane waves with similar directions of propagation. Some recent advances [1] in the general setting of Frame Theory have demonstrated that ill-conditioning arising from such redundancy can be successfully overcome using regularization techniques, provided (this is the key point) there exist accurate approximations in the form of expansions with bounded coefficients. It turns out that PPW approximations with bounded coefficients do not always exist.

To explain this, let us introduce the *circular waves* $b_p(\mathbf{x}) := \beta_p J_p(\kappa r)e^{\imath p\theta}$, for $\mathbf{x} = (r, \theta) \in B_1$ and $p \in \mathbb{Z}$, which are the bounded solutions that are separable in polar coordinates. The normalization coefficients β_p, computed e.g. for a $H^1(B_1)$-norm, grow super-exponentially with $|p|$ and allow to make the family $\{b_p\}_p$ a Hilbert basis for the space of Helmholtz solutions in the unit disk. It can be shown that an approximation $\tilde{b}_p$ of b_p consisting in a finite sum of PPW with a vector of coefficients denoted $\boldsymbol{\mu}$, that is accurate, namely $\|b_p - \tilde{b}_p\| \leq \eta$ for some tolerance $1 \geq \eta > 0$, has necessarily coefficients that satisfy $\|\boldsymbol{\mu}\|_{\ell^1} \geq (1 - \eta)\beta_p$. Owing to the super-exponential growth of β_p with $|p|$, this is a clear example of how accuracy and stability properties (in the sense of bounded coefficients) are sometimes mutually exclusive.

This issue can be understood (and proved) from the *Jacobi–Anger identity*: for any $\mathbf{x} \in B_1$ and $\varphi \in [0, 2\pi)$,

$$e^{\imath\kappa\mathbf{d}(\varphi)\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} \left(\imath^p e^{-\imath p\varphi} \beta_p^{-1}\right) b_p(\mathbf{x}). \qquad (1)$$

The modulus of the coefficients in the above modal expansion as a function of p is reported in Figure 1 (case $\zeta = 0$). This quantity, which is independent of φ, decays super-exponentially away from the 'propagative' modes $|p| \leq \kappa$. The direct implication is that PPW superpositions need cancellation (i.e. subtraction of values numerically close to each other) and large coefficients to approximate Helmholtz solutions with a high-frequency modal content (large $|p|$).

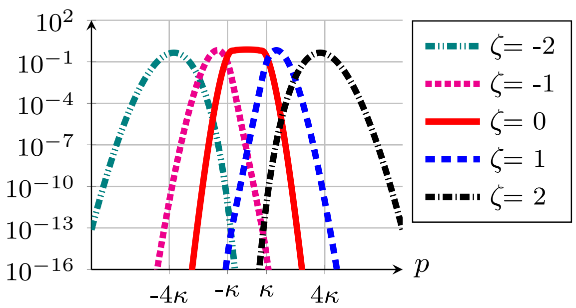

Figure 1: Modulus of the coefficients of the modal expansion in the basis $\{b_p\}_{p\in\mathbb{Z}}$ of various plane waves for $\kappa = 16$.

3 Evanescent plane waves

To obtain accurate expansions with bounded coefficients, we enrich the PPW set with some *evanescent plane waves* (EPW), a technique already used in the Wave Based Method [3]. EPW are characterized by complex-valued direction vector $\mathbf{d}(\varphi, \zeta) := (\cos(\varphi + \imath\zeta), \sin(\varphi + \imath\zeta)) \in \mathbb{C}^2$, where we introduced the evanescence parameter $\zeta \in \mathbb{R}$. The Helmholtz equation is still satisfied since $\mathbf{d} \cdot \mathbf{d} = 1$. The Jacobi–Anger expansion (1) extends to complex $\mathbf{d}$: for any $\mathbf{x} \in B_1$ and $(\varphi, \zeta) \in [0, 2\pi) \times \mathbb{R}$,

$$e^{\imath\kappa\mathbf{d}(\varphi,\zeta)\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} \left(\imath^p e^{-\imath p\varphi} e^{p\zeta} \beta_p^{-1}\right) b_p(\mathbf{x}). \qquad (2)$$

The modulus of the coefficients in the above modal expansion are reported (with a convenient normalization depending only on ζ) in Figure 1 as a function of p. We see that by tuning the evanescence parameter ζ we are able to shift the modal content of the plane waves to higher frequency regimes. As a result, we expect the EPW to be able to capture well the high-frequency modes of Helmholtz solutions.

The main theoretical result that highlights the benefit of EPW is the following.

Theorem 1 *Let $w(\zeta) := e^{-\kappa \sinh|\zeta| + |\zeta|/4}$. For any Helmholtz solution u in the unit disk there exists a unique Herglotz density v that belongs to a proper subspace of the w^2-weighted L^2-space on $[0, 2\pi] \times \mathbb{R}$ such that: for any $\mathbf{x} \in B_1$*

$$u(\mathbf{x}) = \int_{-\infty}^{+\infty} \int_0^{2\pi} v(\varphi, \zeta) e^{\imath\kappa\mathbf{d}(\varphi,\zeta)\cdot\mathbf{x}}\, w^2(\zeta) \mathrm{d}\varphi \mathrm{d}\zeta.$$

Moreover, the operator that maps u to v is invertible and continuous.

This integral representation (that uses EPW) can be seen as a generalization of the classical *Herglotz representation* (that uses only PPW). While only very regular Helmholtz solutions admit a standard Herglotz representation (with density in $L^2([0, 2\pi])$), the generalized Herglotz representation of Theorem 1 holds for any solution. The price to pay for this result is the need for a two-dimensional parameter domain in place of a one-dimensional one. In view of practical implementations, the difficulty is then to construct discrete counterparts of such continuous representations. We propose a procedure to construct suitable finite sets of EPW, that are reasonable in size, using sampling strategies in the parametric domain [2]. Numerical evidence shows that the resulting discrete expansions are both controllably accurate and with bounded coefficients, hence numerically stable. The proof of Theorem 1 and more details can be found in *arXiv:2202.05658*. The extension to other geometries is challenging and ongoing.

References

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