Nonlinear Helmholtz equations with sign-changing diffusion coefficient

Rainer Mandel[1], **Zoïs Moitier**[1,*], **Barbara Verfürth**[2]

[1]Institute for Analysis, Karlsruhe Institute of Technology, Karlsruhe, Germany

[2]Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Karlsruhe, Germany

*Email: `zois.moitier@kit.edu`

Abstract

We study nonlinear Helmholtz equations with sign-changing diffusion coefficients on bounded domains. The existence of an orthonormal basis of eigenfunctions is established making use of weak T-coercivity theory. All eigenvalues are proved to be bifurcation points and the bifurcating branches are investigated both theoretically and numerically. In a one-dimensional model example we obtain the existence of infinitely many bifurcating branches that are mutually disjoint, unbounded, and consist of solutions with a fixed nodal pattern. We also extend the numerics to a Drude model.

Keywords: Helmholtz equation; Bifurcation Theory; Sign-changing; T-coercivity.

1 Problem setting

We are interested in the nonlinear Helmholtz equations in dimension $N \in \{1,2,3\}$ of the form:

$$-\operatorname{div}(\sigma(x)\nabla u) - \lambda c(x) u = \kappa(x) u^3, \quad \text{in } \Omega, \quad (1)$$

for $(\lambda, u) \in \mathbb{R} \times \mathrm{H}_0^1(\Omega)$ where Ω is a bounded open domain of $\mathbb{R}^N$, the diffusion coefficient $\sigma \in \mathrm{L}^\infty(\Omega)$ is sign-changing, $0 < c \in \mathrm{L}^\infty(\Omega)$, and $\kappa \in \mathrm{L}^\infty(\Omega)$. Being sign-changing means that the domain Ω is partitioned in two open subdomains $\Omega_\pm$ such that $\overline{\Omega_-} \cup \overline{\Omega_+} = \overline{\Omega}$, $\Omega_- \cap \Omega_+ = \varnothing$, and the function σ is negative on Ω_- and positive on Ω_+. The main goal is to detect nontrivial solutions of Eq. (1) that bifurcate from the trivial solution family $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$.

Equation (1) occurs in the study of time-harmonic wave propagation across an interface between a dielectric and a metamaterial with negative permeability and nonlinear Kerr-type permittivity [3]. Those two effects have been studied separately. The term κu^3 is a classic manifestation of the nonlinear Kerr-type permittivity. In the case of positive diffusion coefficients σ on the whole domain, it is well known that all the eigenvalues give rise to bifurcation branches. The metamaterial property manifests by having σ negative on the subdomain Ω_-. The linear theory dealing with the well-posedness of such problems for right-hand sides $f(x)$ instead of $\kappa(x)\, u^3$ has been studied both analytically and numerically [1]. The main difficulty with the indefinite operator $u \mapsto -\operatorname{div}(\sigma(x)\nabla u)$ is that the standard theory for elliptic boundary value problems based on the Lax-Milgram Lemma does not apply. The (weak) T-coercivity has been introduced to recover a linear theory in the Fredholm sense. We assume that the operator $u \mapsto -\operatorname{div}(\sigma(x)\nabla u)$ is weakly T-coercive, which means that there exists an isomorphism $\mathtt{T} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ and a compact operator $\mathtt{K} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ such that

$$(u, v) \mapsto \int_\Omega \sigma \nabla u \cdot \nabla(\mathtt{T}v) + \int_\Omega \nabla(\mathtt{K}u) \cdot \nabla v$$

is coercive. Being weakly T-coercive depends on the precise shape of the interface $\partial\Omega_- \cap \partial\Omega_+$ and jumps of σ across the interface but, in dimension 1 and 2, it is known to be the case in many settings.

2 Main results

The weak T-coercivity ensured the existence of an orthonormal basis $(\phi_j)_{j\in\mathbb{Z}}$ consisting of eigenfunctions associated to the linear differential operator $u \mapsto -c^{-1}\operatorname{div}(\sigma\nabla u)$. Due to the sign-change of σ the corresponding sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{Z}}$ satisfy $\lambda_j \to \pm\infty$ as $j \to \pm\infty$. Using some recent bifurcation result for strongly indefinite operators needed because of the sign-change of σ, we can show [3]:

Theorem 1 *Each eigenpair (λ_j, ϕ_j) is a bifurcation point of Eq. (1). If λ_j has an odd geometric multiplicity then the connected component $\mathcal{C}_j \subset \mathbb{R} \times \mathrm{H}_0^1(\Omega)$ containing $(\lambda_j, 0)$ satisfies Rabinowitz' alternative:*
(1) $\mathcal{C}_j$ is unbounded in $\mathbb{R} \times \mathrm{H}_0^1(\Omega)$;
(2) $\mathcal{C}_j$ contains another trivial solution $(\lambda_i, 0)$.

In a 1D setting with the subdomain $\Omega_- = (a_-, 0)$ and $\Omega_+ = (0, a_+)$, and piecewise constant functions σ, c, we can strengthen our result. Using the almost explicit expression of the eigenvalues, we can use the distribution of zeros of the eigenfunction to show:

Corollary 2 *The connected component $\mathcal{C}_j \subset \mathbb{R} \times \mathrm{H}^1_0(\Omega)$ is unbounded and $\mathcal{C}_j \cap \mathcal{C}_i = \varnothing$ for $i \neq j$.*

Using additional assumptions, see [3, Theorem 6.3], which has been verified in the 1D settings, we can show a variational result for our problem. Meaning, for a fix $\lambda \in \mathbb{R}$, there exists infinitely many solutions of Eq. (1).

3 Bifurcation visualization

We consider $\Omega_- = (-5, 0)$, $\Omega_+ = (0, 5)$, the diffusion coefficient is chosen piecewise constant $\sigma|_{\Omega_-} \equiv -1.005$ and $\sigma|_{\Omega_+} \equiv 1$, $c \equiv 1$, and $\kappa \equiv 1$. The numerics have been done using a finite element discretization and the Matlab package `pde2path` [4]. The finite element discretization use a `T`-conform mesh which is refined close to the interface $\{0\}$ to faithfully represent the interface behavior. Figure 1 is the 2D bifurcation diagram (abscissa λ and ordinate $\|u\|_{\mathrm{L}^2(\Omega)}$) where λ is initialized in $[-10, 15]$. All branches are unbounded and seemingly do not contain points of secondary bifurcation.

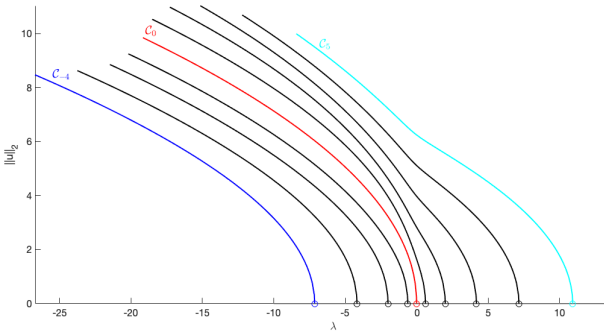

Figure 1: Bifurcation diagram.

4 Extension: The Drude model

A simple known model to have negative diffusion parameter is the Drude model, however in that model the coefficients σ and c depend on the spectral parameter λ. In [2], the diffusion coefficient σ and coefficient c are now given by

$$\sigma_\lambda(x) = \frac{1}{1 - \mathbf{1}_{\Omega_-}(x)\frac{\Lambda_\sigma}{\lambda}}, \quad c_\lambda(x) = 1 - \mathbf{1}_{\Omega_-}(x)\frac{\Lambda_c}{\lambda}$$

where $\mathbf{1}_{\Omega_-}(x)$ is the indicator function of Ω_- and with $0 < \Lambda_c < \Lambda_\sigma$ constants.

For the numerical study, we use the same 1D subdomain $\Omega_- = (-5, 0)$ and $\Omega_+ = (0, 5)$, and $\Lambda_c = 4$, $\Lambda_\sigma = 7$, and $\kappa \equiv 1$. Using the same numerical method as before, Figure 2 we show the 2D bifurcation diagram (abscissa λ and ordinate $\|u\|_{\mathrm{L}^2(\Omega)}$) where λ is initialized in $[0.1, 9]$. We observe that the branches seems to accumulate at $\lambda = 0$ and, around $\lambda = \Lambda_\sigma$, we see a branch that seems to have a vertical asymptote and another that non-smoothly cross the values Λ_σ. We expect more intricate branch behavior around the values $\lambda = 0$ or Λ_σ as either c_λ or Λ_σ blow-up around those points but a more rigorous study is needed.

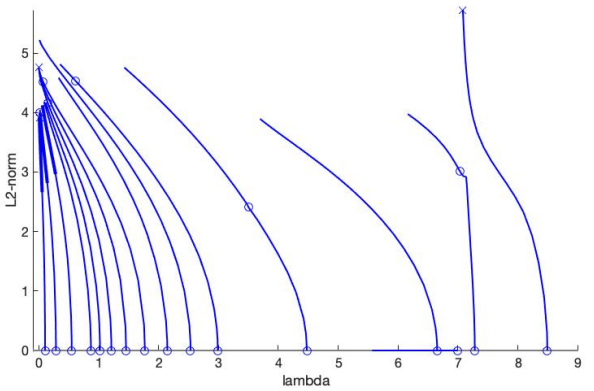

Figure 2: Drude model: bifurcation diagram.

References

[1] A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet, Jr., `T`-coercivity for scalar interface problems between dielectrics and metamaterials, *ESAIM: M2AN* (2012).

[2] C. Hazard, S. Paolantoni, Spectral analysis of polygonal cavities containing a negative-index material, *AHL* (2020).

[3] R. Mandel, Z. Moitier, B. Verfürth, Nonlinear Helmholtz equations with sign-changing diffusion coefficient, *to appear in C. R. Math.*, arXiv:2107.14516.

[4] H. Uecker, D. Wetzel, J. Rademacher, pde2path - a Matlab package for continuation and bifurcation in 2D elliptic systems, *NMTMA* (2014).

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