Sound Propagation in Slowly-Varying Lined Ducts

Tom White[1,*], Raphael Assier[1], William Parnell[1]
[1]Department of Mathematics, University of Manchester, United Kingdom
*Email: thomas.white-4@manchester.ac.uk

Abstract

This talk will consider the modelling of sound propagation in slowly varying ducts. In straight cylindrical ducts, lined with a locally reacting material, sound transmission can be decomposed into a set of modal solutions. However, in practise, the ducts we wish to model have boundaries which vary slowly along the axial direction. In order to maintain semi-analytic modal solutions, the WKB (Wentzel-Kramers-Brillouin) approximation is adopted. The approximation assumes that the boundaries vary sufficiently slowly, that any propagating modes remain in the same eigenstate and do not experience any reflection. This allows for the introduction of a small parameter ϵ, related to the gradient of the duct boundaries, which allows for multiple scale asymptotics. The resulting solutions are slowly varying modes. Such modes have been studied extensively to leading order, however, in this talk we shall consider the first order behaviour in ϵ.

Keywords: aeroacoustics, multiple-scales, impedance

1 Introduction

The study of sound propagation in slowly-varying lined ducts is well established to leading order, with [1] and [2] considering the problem without flow, and in the presence of an axial mean flow respectively. In this talk we shall consider the case of extending the previous work on ducts without flow to higher order in ϵ, with a view to completing the analysis to include mean axial flow in the near future. The ducts are considered to be lined with a locally reacting acoustic material, such as an array of Helmholtz resonators, which can be modelled by a frequency dependent surface impedance. In order to preserve modal analysis, the WKB approximation is assumed, which is that if the geometry is sufficiently slowly varying, any propagating modes will remain in the same eigenstate and will not experience any reflection.

2 Definition of the Problem

Let us consider an axis-symmetric annular duct, defined in polar co-ordinates (r, θ, x), with boundaries whose position's vary in the axial direction. At both boundaries, the duct is lined with a locally reacting acoustic material. Then, we define a slowly varying parameter, ϵ, which is related to the rate of change of the duct boundaries. Here we take ϵ to be the root mean square of the gradient of the boundaries in x. Note, that ϵ is not present in the final solutions, so can be chosen to be any measure of the speed of variance of the boundaries.

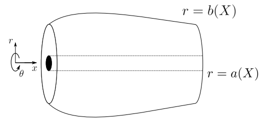

Figure 1: Sketch of an annular duct with slowly varying boundaries.

A slowly varying co-ordinate, $X = \epsilon x$, is defined, which the slowly varying boundaries, $r = a(X)$ and $r = b(X)$, depend on. After assuming periodic solutions in time t and angle θ and performing the multiple scales analysis outlined in [3], an ansatz of the following form is posed:

$$\begin{aligned} P_m(r, X) = \big(&\tilde{p}_{m,0}(r, X) + \epsilon\tilde{p}_{m,1}(r, X) \\ &+\epsilon^2\tilde{p}_{m,2}(r, X)\big) \exp\Big\{-\frac{i}{\epsilon}\int^X k_0(X')dX'\Big\}, \end{aligned} \tag{1}$$

where $P_m(r, X)$ is a single time harmonic mode, m is the azimuthal order and the axial periodicity is governed by the slowly wavenumber $k_0(X)$. Substituting (1) into the Helmholtz equation and collecting powers of ϵ gives the hierarchy of equations:

$$\begin{aligned} \mathcal{O}(1) :&\mathcal{L}_0(\tilde{p}_{m,0}) = 0, \\ \mathcal{O}(\epsilon) :&\mathcal{L}_0(\tilde{p}_{m,1}) = \mathcal{F}_0(\tilde{p}_{m,0}), \\ \mathcal{O}(\epsilon^2) :&\mathcal{L}_0(\tilde{p}_{m,2}) = \mathcal{F}_0(\tilde{p}_{m,1}) + \mathcal{F}_1(\tilde{p}_{m,0}), \end{aligned} \tag{2}$$

where $\mathcal{L}_0$ is the 2nd order linear differential operator for Bessel's equation, and $\mathcal{F}_{0,1}$ are known forcing terms. For the boundary condition, it is assumed that at $r = a(X)$, $b(X)$ the surface impedance are given by $Z_a(\omega)$ and $Z_b(\omega)$ respectively. This leads to the boundary conditions:

$$\frac{\partial P_m}{\partial \mathbf{n}_{a,b}} = \frac{\omega P_m}{i Z_{a,b}} \text{ at } r = a(X),\ b(X), \qquad (3)$$

where $\mathbf{n}_{a,b}$ is the normal pointing into the boundary. The hierarchy of equations for the boundary conditions are derived by expanding the normal at each boundary in powers of ϵ.

3 Finding Modal Solutions

The leading order solution is well established to be a weighted sum of Bessel functions of the first kind, multiplied by a slowly varying amplitude,

$$\tilde{p}_{m,0} = A_0(X)\big(J_m(\alpha_0(X)r) + \Xi(X)Y_m(\alpha_0(X)r)\big), \quad (4)$$

where $\alpha_0(X)$ is the, slowly varying, radial wave number and $\Xi(X)$ is known. In order to find the slowly varying amplitude $A_0(X)$, the $\mathcal{O}(\epsilon)$ governing equations must be considered. It is not necessary to consider the full solution to this problem, as we can apply a solvability condition, by considering Green's second identity:

$$\begin{aligned}\int_{a(X)}^{b(X)} \big(\tilde{p}_{m,0}\mathcal{L}_0(\tilde{p}_{m,1}) - \tilde{p}_{m,1}\mathcal{L}_0(\tilde{p}_{m,0})\big) r' dr' \\ = \left[\tilde{p}_{m,0}\frac{\partial \tilde{p}_{m,1}}{\partial r} - \frac{\partial \tilde{p}_{m,0}}{\partial r}\tilde{p}_{m,1}\right]_{r=a(X)}^{b(X)}.\end{aligned} \qquad (5)$$

Upon substitution of the governing equations and boundary conditions, (5) can be reduced to find an explicit form of $A_0(X)$. However, in the case considered, the full solution to the $\mathcal{O}(\epsilon)$ equations is required. One way to find this solution is to apply the method of variation of parameters to the inhomogeneous ODE. By applying the boundary conditions we find $A_0(X)$ to be the same as applying (5), but leaves a similar unknown slowly varying coefficient $A_1(X)$ at first order. In order to solve for $A_1(X)$ another solvability condition can be applied, this time on the $\mathcal{O}(\epsilon^2)$ equations, which leads to an ODE to solve for $A_1(X)$ of the form

$$\mathfrak{A}(X)\frac{\mathrm{d}A_1}{\mathrm{d}X} + \mathfrak{B}(X)A_1(X) + \mathfrak{C}(X) = 0, \qquad (6)$$

where $\mathfrak{A}(X)$, $\mathfrak{B}(X)$ and $\mathfrak{C}(X)$ are known functions.

4 Results and Comparisons

To demonstrate the improved accuracy of the first order solution, compared to the leading order, we consider the example of cylindrical duct with a cosine outer boundary. The boundary is chosen, such that $\epsilon = 0.1$. The angular frequency is $\omega = 3$, and $m = 0$, with surface impedance $Z_b = 2 - i$.

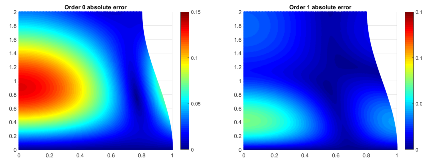

Figure 2: The absolute error between the complex pressure fields for the leading and first order propagating modes, compared with numeric results.

Figure 2 shows the absolute error between the pressure fields, $P_m(r, X)$, at $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ and numeric simulations performed using COMSOL. The numeric solution for a single mode is found by applying the mode shape as a boundary condition at $x = 0$. The geometry shown is an axis-symmetric slice in (r, x). Clearly it can be seen that the first order correction provides a significant reduction in error between the fields.

5 Flow Duct Extension

Depending on whether or not results are generated in time for the conference, the first order solution could also be presented for the case of axial mean flow, as this is the ultimate goal of the current work. The method of solution will be the same, however, the inhomogenous parts of the equations becomes more complex when flow is introduced, leading to greater computational complexity.

References

[1] Nayfeh, A.H. and Telionis, D.P., Acoustic propagation in ducts with varying cross sections, *The Journal of the Acoustical Society of America* **54(6)** (1973), pp.1654–1661.

[2] Rienstra, S.W., Sound transmission in slowly varying circular and annular lined ducts with flow, *Journal of Fluid Mechanics,* **380** (1999) pp.279–296.

[3] Nayfeh, A.H., *Perturbation Methods*, John Wiley & Sons, 2008.