The Half-space Matching method and the Perfectly Matched Layers for scattering problem in anisotropic elastic media

E. Bécache[1], A.-S. Bonnet-Ben Dhia[1], S. Fliss[1], A. Tonnoir[2,*]
[1]POEMS(UMR 7231 CNRS-ENSTA-INRIA, Palaiseau, France)
[2]LMI (EA 3226 - FR CNRS 3335), Rouen, France
*Email: antoine.tonnoir@insa-rouen.fr

Abstract

In this work, we are interested in solving scattering problems in anisotropic elastic unbounded domains. We propose an extension of a new method called the Half-Space Matching (HSM) method which enables to consider any anisotropy. Also, we compare HSM results with Perfectly Matched Layers (PMLs) results for several anisotropic media to emphasize the robustness of the HSM method.

Keywords: Anisotropic elastodynamics, unbounded domains, Domain decomposition methods.

1 Introduction

Elastodynamics scattering problems occur for instance in the context of geophysical surveys or non destructive testing simulations. Classically, the difficulty is to reformulate the problem in a bounded domain to solve it numerically. Several approaches exist in the literature, such as absorbing layers, absorbing or transparent boundary conditions or integral equation methods, but they usually cannot handle every anisotropy. In particular, it is well-known for time-domain regime [1] that the PMLs method suffers from instabilities for some anisotropic materials. For frequency-domain, to the best of our knowledge, fewer results exist and the equivalent of the instabilities is less clear.

To consider general anisotropic materials, we propose an extension of the HSM method, first introduced for anisotropic scalar equations [2], to the elastodynamic case. Also, we believe that the HSM formulation can help to understand some curious results observed using PMLs in the frequency-domain for elastic media. A comparison study between the HSM and the PMLs methods for various anisotropic materials will also be shown.

2 The HSM formulation

We consider the scattering problem:

$$\left| \begin{array}{lll} -\mathrm{div}\sigma(\mathbf{u}) - \rho\omega^2\mathbf{u} = 0 & \text{in} & \Omega = \mathbb{R}^2 \setminus \mathcal{O}, \\ \sigma(\mathbf{u})\nu = \mathbf{g} & \text{on} & \partial\mathcal{O}, \end{array} \right. \tag{1}$$

where $\mathbf{u}$ is the diffracted field, $\mathcal{O}$ is a bounded obstacle, ν is the unit outward normal and the stress tensor $\sigma(\cdot)$ is linked to the strain tensor $\varepsilon(\cdot)$ via the general anisotropic Hooke's law (using Voigt's notations):

$$\begin{bmatrix} \sigma_{xx}(\mathbf{u}) \\ \sigma_{yy}(\mathbf{u}) \\ \sigma_{xy}(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}(\mathbf{u}) \\ \varepsilon_{yy}(\mathbf{u}) \\ 2\varepsilon_{xy}(\mathbf{u}) \end{bmatrix}.$$

The coefficients C_{ij} are supposed to be local perturbation of constant coefficients as well as the density ρ.

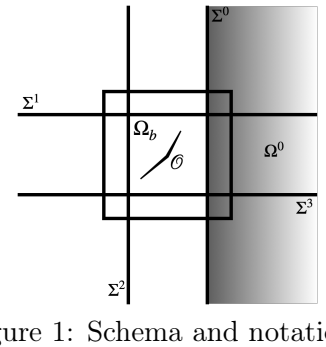

Figure 1: Schema and notations.

To get the HSM formulation of this problem, the idea is to split the domain into five parts, see Figure 1:

- a square $\Omega_b = [-b, b]^2 \setminus \mathcal{O}$, $b > 0$, containing all the heterogeneities and in which we use a finite elements representation of the solution denoted by $\mathbf{u}^b$,
- and four half-spaces Ω^j in which we use Fourier-integral representations of the solution denoted by $\mathbf{u}^j$.

More precisely, taking advantage of the homogeneity of the medium in the half-spaces, we can

use the Fourier transform in the transverse direction to express the solution $\mathbf{u}^j$ in Ω^j as a function of its trace $\mathbf{\Phi}^j$ on $\Sigma^j := \partial\Omega^j$. For instance, in $\Omega^0 = \{x \geq a\} \times \mathbb{R}$, $0 < a < b$, we get an expression of the form:

$$\mathbf{u}^0(x,y) = \int_{\mathbb{R}} Q(\xi) e^{\mathrm{i}K(\xi)(x-a)+\mathrm{i}\xi y} Q^{-1}(\xi) \widehat{\mathbf{\Phi}}^0(\xi) \mathrm{d}\xi \tag{2}$$

where $Q(\xi)$ is a 2×2 matrix and $K(\xi)$ is a 2×2 diagonal matrix that will be precised in the talk. As we can notice, this representation is similar to plane waves representation, with the difference that it also takes into account evanescent plane waves (when $\mathrm{Re}(K(\xi)) \neq 0$). Let us simply add that to get these expressions in each half-space, we need to properly select outgoing waves based on the direction of their group velocity.

Then, to ensure the matching between these different representations in the different subdomains, we must impose compatibility conditions on the boundaries of each subdomain. This leads to a formulation where the new unknowns are $\mathbf{u}^b$ in the Ω_b and the traces $\mathbf{\Phi}^j$ on Σ^j which are coupled via integral operators. This HSM formulation is suitable for discretization and has been validated in the isotropic case by comparison with an analytical reference solution. As already mentioned, it is also suitable for anisotropic media, as illustrated on Figure 2, and we can make an a posteriori reconstruction of the solution in the four half-spaces discretizing formula (2).

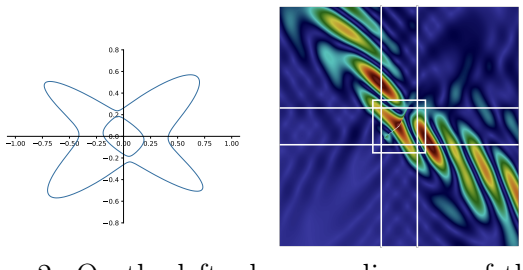

Figure 2: On the left, slowness diagram of the anisotropic material. On the right, modulus of the diffracted field in Ω_b and reconstruction in Ω^j.

3 The PMLs method

In the time-harmonic regime, the PMLs formulation is obtained by using a complex scaling in the exterior of the domain of interest. In particular, we can consider the simple change of variables $\widetilde{t} = \alpha(t)(t-b)+b$, $t \in \{x, y\}$, where $\alpha(t) = 1$ if $|t| \leq b$ and $\alpha(t) = me^{i\theta}$ if $|t| > b$ with $m > 0$ and $\theta \in [0, \frac{\pi}{2}[$. Then, we expect the PML solution to exponentially decay in the layers, so that we can truncate them at a finite distance.

Now, using the half-space representation (2), we easily show that the PML solution in Ω^0 (considering only stretching in direction x) is exponentially decaying if $\mathrm{Im}(e^{i\theta}K(\xi)) > 0$, $\forall \xi \in \mathbb{R}$. One interesting feature is the fact that this condition concerns both propagative and evanescent plane waves. In particular, in presence of backward waves, which corresponds to the case where $\mathrm{Im}(K(\xi)) = 0$ and $\mathrm{Re}(K(\xi)) < 0$ for ξ in an interval, this condition can never be satisfied (let us mention that this case leads to instabilities in time-domain). Also, for evanescent waves when $\mathrm{Im}(K(\xi)) > 0$, if $\mathrm{Re}(K(\xi)) \neq 0$ (corresponding to inhomogeneous waves) this condition imposes that $\theta \leq \theta^*$ with $\theta^* < \frac{\pi}{2}$. Numerically, if this condition for evanescent waves is not satisfied, even in absence of backward waves, we can observe a strange behavior of the solution in the PMLs as illustrated in Figure 3. We therefore guess that these inhomogeneous plane waves are important to understand the behavior of the PML solution. In the talk, several numerical tests will be shown to study this question.

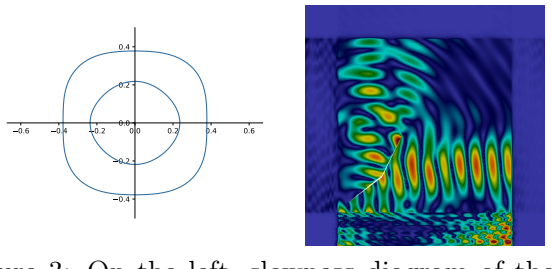

Figure 3: On the left, slowness diagram of the orthotropic material. On the right, modulus of the diffracted field in the physical domain and in the PMLs.

References

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