Instability and over-reflection of acoustic waves in compressible boundary layer flows

Felician Putz[1,*], **Yi Zhang**[1], **Martin Oberlack**[1]
[1]Chair of Fluid Dynamics, TU Darmstadt, Darmstadt, Germany
*Email: putz@fdy.tu-darmstadt.de

Abstract

2D direct numerical simulations of acoustic waves in supersonic compressible boundary layer flows were conducted using the high order discontinuous Galerkin method. The purpose was to verify both theoretically predicted instabilities and to show two mode doubling effects caused by a doubling in frequency and wave number, respectively. Secondly, simulated over-reflection coefficients show good agreements to theoretical results, and, most important, resonant over-reflection is verified in a narrow frequency band.

Keywords: PBE, over-reflection, DNS, stability, boundary layer acoustics

1 Introduction

Acoustic waves play a key role in numerous engineering applications. When these waves interact with a mean shear flow, e.g. in a boundary layer, the propagation is significantly affected and effects such as instabilities can arise so that the amplitude of an unstable wave increases with time. This may lead to a transition to turbulence and hence must be well understood.

Another acoustic effect in boundary layers is the reflection at the surface. During this process an incident wave traveling through the large velocity gradients in the shear layer deforms in amplitude and shape. Therefore over-reflection may occur which is when the amplitude of the reflected wave is larger than the amplitude of the incident wave. In this case, the acoustic wave absorbs energy from the main flow. The strength of this effects varies with wave number and frequency of the incident wave and in particular resonant over-reflection may occur in the vicinity of unstable modes, which is accompanied by a strong exaggeration of the amplitude.

2D direct numerical simulations (DNS) are conducted to study temporal instability and over-reflection using the high order discontinuous Galerkin (dG) method.

2 Boundary layer acoustics

This study is confined to compressible, inviscid, homentropic boundary layer flows with an exponential velocity profile as a model problem. Based on these assumptions, the Euler equations can be linearized in terms of small acoustic perturbations. Combined into one equation and non-dimensionalized with the boundary layer thickness, the free-stream velocity and the density of the free-stream, this yields the Pridmore-Brown equation

$$\begin{aligned}&\frac{\mathrm{d}^2\hat{\rho}}{\mathrm{d}y^2}+\frac{2\alpha\mathrm{e}^{-y}}{\omega-\alpha+\alpha\mathrm{e}^{-y}}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}y}\\&+\left[\mathrm{M}^2\left(\omega-\alpha+\alpha\mathrm{e}^{-y}\right)^2-\alpha^2\right]\hat{\rho}=0.\end{aligned}\tag{1}$$

This second order ODE for the density amplitude $\hat{\rho}$ defines both the propagation and stability of the acoustic wave. Zhang & Oberlack [1] derived an exact solution to eq. (1) in terms of confluent Heun functions denoted by $Hc\left(\cdot;\frac{\alpha}{\alpha-\omega}\right)$, in which "$\cdot$" is an abbreviation for parameters depending on the dimensionless wave number α, dimensionless frequency ω and Mach number M [1]. The solution contains two Heun functions Hc and $\widetilde{Hc}$ with different parameters that represent the incident and the outgoing wave. Based on this, the over-reflection coefficient is defined by

$$R=\frac{\left(\mathrm{iM}\alpha+\sqrt{\theta}\right)\widetilde{Hc}\left(\cdot;\frac{\alpha}{\alpha-\omega}\right)+\frac{\alpha}{\alpha-\omega}\widetilde{Hc}'\left(\cdot;\frac{\alpha}{\alpha-\omega}\right)}{\left(-\mathrm{iM}\alpha+\sqrt{\theta}\right)Hc\left(\cdot;\frac{\alpha}{\alpha-\omega}\right)-\frac{\alpha}{\alpha-\omega}Hc'\left(\cdot;\frac{\alpha}{\alpha-\omega}\right)}\tag{2}$$

with the parameter θ depending on α, ω and M. R is the ratio between the amplitude of the reflected to the incident wave. It is calculated from the solution to eq. (1) using the boundary conditions of an unit amplitude incident wave and vanishing normal velocity at the wall.

Stability is investigated by only considering a vanishing amplitude $\hat{\rho}$ as $y\to\infty$. This is equivalent to the special case of $R=0$ in which the numerator of eq. (2) defines an eigenvalue equation for the eigenvalue ω [1].

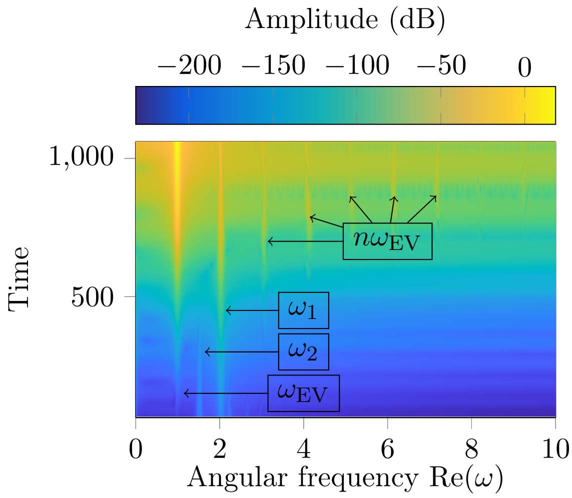

Figure 1: Short time Fourier transformation of the difference between the analytical solution of the pressure and the DNS. $\mathrm{M} = 5$ and $\alpha = 1.5$. The frequency given by the eigenvalue problem is $\omega_{\mathrm{EV}} = 1.028 + \mathrm{i}1.0770 \cdot 10^{-2}$.

3 Temporal instability

The above mentioned eigenvalue equation has at least one solution for each Mach and wave number. The eigenvalue is written as $\omega_{\mathrm{EV}} = \omega(\mathrm{M}, \alpha)$ and is temporarily unstable for $\mathrm{Im}(\omega_{\mathrm{EV}}) > 0$. Having this in mind, a DNS is conducted with the non-linear Euler equations using the dG method of order 6. The initial condition is the incident wave of the linearized solution of eq. (1). Comparison of DNS and theory shows good agreement between the simulation as long as the amplitude of the pressure perturbation is less than 3% of the base flow pressure [2]. A deeper analysis reveals that the error caused by the linearization are due to two different mode doubling effects. A mode with twice the eigenvalue frequency $\omega_1 = 2\omega_{\mathrm{EV}}$ and a mode, whose frequency $\omega_2 = \omega(\mathrm{M}, 2\alpha)$ is the solution to the eigenvalue problem with a doubling of the wave number, are present. The two mode doubling frequencies are shown in figure 1 at $\omega_1 \approx 2$ and $\omega_2 \approx 1.5$. Additional modes for $n > 2$ are present but their amplitude is negligible compared to $n = 1$. The growth rates differ for each mode and for long times the amplitude at the original eigenvalue is dominant and the mode doubling effects become negligible.

4 Over-reflection

Simulations of over-reflection are conducted by placing an incident wave packet in the unsheared region of the domain. As R is sensitive towards

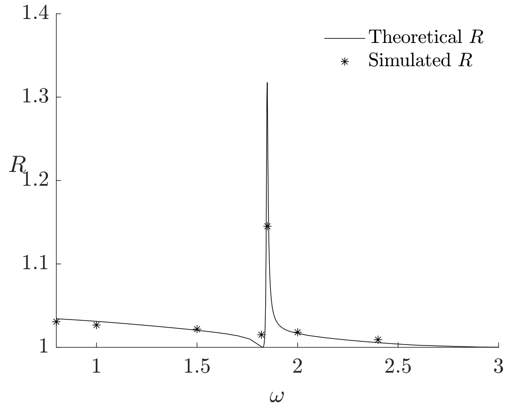

Figure 2: Comparison between theoretical reflection coefficients and simulated results for $\mathrm{M} = 5$, $\alpha = 4$ and different frequencies ω. Simulated data is taken from [3].

variations of wave number or frequency, particular attention was paid to the generation of the wave packet so that the wave is not distorted. The wave packet is then reflected at the wall and the ratio of the amplitude of the reflected wave to the incident wave is the over-reflection coefficient R, see eq. (2). The coefficient is extracted from the simulation by comparing the amplitude of incident and reflected waves over a section outside of the boundary layer. The result in comparison with the analytical values for R computed by Zhang et. al (2022) are shown in figure 2 for $\mathrm{M} = 5$, $\alpha = 4$ and varying frequencies [3]. Over-reflection occurs for all frequencies and a sharp peak, the resonant over-reflection, is located at $\omega \approx 1.85$. This is the frequency of the eigenvalue solution (numerator of eq. (2)) for the Mach and wave number. The simulation shows good agreements with the theoretical values of R. In particular, the resonant over-reflection is captured well.

References

[1] Zhang, Y. & Oberlack, M. Inviscid instability of compressible exponential boundary layer flows. *AIP Advances*. **11**, 105308 (2021,10)

[2] Putz, F. Direct Numerical Simulation of Linear and Non-linear Acoustic Instabilities in Boundary Layer Flow. Master's Thesis. TU Darmstadt. (2021)

[3] Zhang, Y., Görtz, S. & Oberlack, M. Over-reflection of acoustic waves by compressible boundary layer flows. *Journal of Fluid Mechanics (under review)*. (2022)