Numerical treatment of the vectorial equations of stellar oscillations

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Abstract

We present a discretization for the vectorial equations of solar and stellar oscillations. Special attention is paid to preserving compatibility with the generalized Helmholtz decomposition used in the analysis of the continuous model to achieve stability.

Keywords: finite element method, Galbrun's equation, helioseismology

1 Introduction

The Galbrun's equation with additional rotational and gravitational terms model stellar oscillations. Recently, HDG numerical methods for the related scalar case, a convected Helmholtz equations, have been devised and analysed in [1]. Furthermore, in [2], it was shown that the vector valued problem is well-posed, when incorporating a simple damping term. A suitable generalized Helmholtz decomposition plays a crucial role in the analysis. In the discretization, we aim to preserve a discrete version of the generalized Helmholtz decomposition, which is crucial for stability and helpful for the numerical analysis. We present an $H(\mathrm{div})$-conforming numerical method that respects the structural properties of the continuous problem and introduce the tools needed for the numerical analysis.

2 Setting

Galbrun's equation for time-harmonic acoustic waves for the unknowns $\mathbf{u}, \psi$ is given by the partial differential equation

$$\begin{aligned}
&\rho(-i\omega + (\mathbf{b}\cdot\nabla))^2\mathbf{u} - \nabla(\rho c_s^2 \nabla\cdot \mathbf{u})\\
&+(\nabla\cdot \mathbf{u})\nabla p - \nabla(\nabla p\cdot \mathbf{u}) - i\omega\gamma\rho\mathbf{u}\\
&+(\mathrm{Hess}(p) - \rho\,\mathrm{Hess}(\phi))\mathbf{u} + \rho\nabla\psi = \mathbf{f}\\
&\qquad -\frac{1}{4\pi G}\Delta\psi + \nabla\cdot(\rho\mathbf{u}) = 0.
\end{aligned} \tag{1}$$

in the presence of density ρ, pressure p, sound speed c_s, background velocity $\mathbf{b}$, gravitational background potential ϕ, damping coefficient γ, gravitational constant G, and source $\mathbf{f}$. This problem was shown to be well-posed in [2]. For the discretization, we will focus on a common simplification of the problem, the Cowling approximation, given by setting $\psi = 0$ in (1). We consider the resulting variational problem over the Hilbert space

$$\begin{aligned}
\mathbb{X}_{\mathbf{b}} = \{\mathbf{u} \in \mathbf{L}^2(\Omega, \mathbb{C}^3):\ &\nabla\cdot \mathbf{u} \in L^2(\Omega),\\
&(\mathbf{b}\cdot\nabla)\mathbf{u} \in \mathbf{L}^2(\Omega,\mathbb{C}^3),\ \mathbf{u}\cdot n_{\mathbf{x}} = 0 \text{ on } \partial\Omega\}
\end{aligned}$$

over a bounded domain Ω and with inner product

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}_{\mathbf{b}}} = \langle \mathbf{u}, \mathbf{u}'\rangle + \langle \nabla\cdot \mathbf{u}, \nabla\cdot \mathbf{u}'\rangle\\
+ \langle (\mathbf{b}\cdot\nabla)\mathbf{u}, (\mathbf{b}\cdot\nabla)\mathbf{u}'\rangle.
\end{aligned}$$

The variational problem states: Find $\mathbf{u} \in \mathbb{X}_{\mathbf{b}}$ such that

$$a(\mathbf{u}, \mathbf{u}') = \sum_{i=1}^{3} a^i(\mathbf{u}, \mathbf{u}') = \langle \mathbf{f}, \mathbf{u}'\rangle,\ \forall \mathbf{u}' \in \mathbb{X}_{\mathbf{b}},$$

with

$$\begin{aligned}
a^1(\mathbf{u}, \mathbf{u}') &= \langle \rho c_s^2(\nabla + \mathbf{q})\cdot \mathbf{u}, (\nabla + \mathbf{q})\cdot \mathbf{u}'\rangle\\
&\quad - \langle \rho c_s^2 \mathbf{q}\cdot \mathbf{u}, \mathbf{q}\cdot \mathbf{u}'\rangle\\
a^2(\mathbf{u}, \mathbf{u}') &= \langle (\mathrm{Hess}(p) - \rho\,\mathrm{Hess}(\phi))\mathbf{u}, \mathbf{u}'\rangle\\
&\quad - \langle \rho\mathcal{D}\mathbf{u}, \mathcal{D}\mathbf{u}'\rangle\\
a^3(\mathbf{u}, \mathbf{u}') &= -i\rho\gamma\omega\langle \mathbf{u}, \mathbf{u}'\rangle,
\end{aligned}$$

where we have introduced the operator $\mathcal{D} := (-i\omega + (\mathbf{b}\cdot\nabla))$ modeling transport and the abbreviation $\mathbf{q} := \rho^{-1} c_s^{-2}\nabla p$.

We review the essential parts of the proof relevant for the analysis of the discretization. Note that $\mathbb{X}_{\mathbf{b}}$ is not compactly embedded in $\mathbf{L}^2$ and thus we cannot derive Fredholmness using a Gårding-type inequality.

First, a generalized version of a Helmholtz decomposition for the space $\mathbb{X}_{\mathbf{b}}$, presented in [2, Lemma 3.5], given by the decomposition into gradient fields, a generalization of divergence-free fields, and a finite dimensional space, i.e.

$$\mathbb{X}_{\mathbf{b}} = \mathbf{V} \oplus \mathbf{W} \oplus \mathbf{Z}$$

with $\mathbf{V} \subset \{\nabla v \mid v \in H^2(\Omega),\ \nabla v \cdot n_{\mathbf{x}} = 0 \text{ on } \partial\Omega\}$, $\mathbf{W} = \{\mathbf{u} \in \mathbb{X}_{\mathbf{b}} \mid (\nabla + \mathbf{q}) \cdot \mathbf{u} = 0\}$ and $\mathbf{Z}$ with size related to $\ker(B)$, with $B \in L(H^1)$, $\langle Bu, v\rangle := \langle \nabla u, \nabla v\rangle - \langle \mathbf{q} \cdot \nabla u, v\rangle$. Second, the operator A, associated to the bilinear form $a(\mathbf{u}, \mathbf{u}') = \langle A\mathbf{u}, \mathbf{u}'\rangle_X$ is shown to be weak T-coercive, i.e.

$$AT = B^{\text{coercive}} + B^{\text{compact}}$$

for a coercive and a compact operator B^{coercive}, B^{compact}. Injectivity of A is caused by the damping term. Well-posedness follows using Fredholm alternative. The choice of T, which we will also use in the discrete setting, is a sign switch operator: Let us denote by $P_{\mathbf{V}}, P_{\mathbf{W}}, P_{\mathbf{Z}}$ the respective projections, then T is chosen as

$$T := P_{\mathbf{V}} - P_{\mathbf{W}} + P_{\mathbf{Z}}. \tag{2}$$

3 Discretization

We follow [2], by considering two distinct cases for discretization, first focusing on the case with only background flow present without pressure and gravity and then the contrariwise case, which we later merge.

To approximate $\mathbf{u}$ we consider discrete functions $\mathbf{u}_h \in \mathbb{X}_h$ on which we only impose normal continuity, i.e. $\mathbb{X}_h$ is taken to be a $\mathbf{H}_0(\nabla\cdot;\Omega)$-conforming finite element space, as we cannot expect functions in $\mathbb{X}_{\mathbf{b}}$ to be tangential continuous. Finding a generalized Helmholtz decomposition for the discretization space, similar to the one in the continuous setting, will be crucial for the stability analysis. We aim for an implicit decomposition of the form

$$\mathbb{X}_h = \mathbf{V}_h \oplus \mathbf{W}_h,$$

here we will focus on the crucial spaces $\mathbf{V}, \mathbf{W}$, assuming that the finite dimensional space $\mathbf{Z}$ only contains the zero function.

3.1 Background flow

We first consider the case of constant pressure and gravitational potential ($p = \text{const}$ and $\phi = \text{const}$). We obtain an orthogonal decomposition using the spaces $\mathbf{W}_h = \{\mathbf{u} \in \mathbb{X}_h \mid \nabla\cdot \mathbf{u} = 0\}$ and $\mathbf{V}_h = \{\mathbf{v}_h \in \mathbb{X}_h \mid \langle \mathbf{v}_h, \mathbf{w}_h\rangle_{1,h} = 0,\ \forall \mathbf{w}_h \in \mathbf{W}_h\}$. We introduce the discrete bilinear form

$$a_h(\mathbf{u}_h, \mathbf{u}_h') = a^1(\mathbf{v}_h, \mathbf{v}_h') + a_h^2(\mathbf{u}_h, \mathbf{u}_h') + a^3(\mathbf{u}_h, \mathbf{u}_h')$$

where $a_h^2(\cdot,\cdot)$ has additional DG penalization terms for the tangential jumps across interelement boundaries. Note that since $\mathbf{u}_h = \mathbf{v}_h + \mathbf{w}_h$, with $\mathbf{w}_h$ divergence free, the term $a^1(\cdot,\cdot)$ only depends on functions in $\mathbf{V}_h$.

To show well-posedness we make use of a similar sign switch operator as in the continuous case $T_h := P_{\mathbf{V}_h} - P_{\mathbf{W}_h}$. Crucial for the numerical analysis is the observation that the divergence term can dominate the broken H^1-norm, which gives us control over the other $\mathbf{v}_h$ terms in the bilinear form. Indeed, using tools from the numerical analysis of Stokes and linear elasticity problems for the space $\mathbb{X}_h$, we obtain

$$\|\nabla\cdot \mathbf{v}_h\|_{0,h} \geq c\|\mathbf{v}_h\|_{1,h},\ \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3}$$

To prove convergence we derive a type of commutative property of the sign switching operators

$$\lim_{h\to 0} \|T_h p_h u - p_h T u\|_{\mathbb{X}_h} = 0,\ \forall u \in \mathbb{X}_{\mathbf{b}},$$

where p_h is the projection onto $\mathbb{X}_h$.

3.2 Pressure and gravity

Next we consider the case with no flow ($\mathbf{b} = 0$), but with non-constant pressure and gravity. We must weaken the condition in $\mathbf{W}_h$, else the discrete space may collapse to only containing the trivial solution. We consider $\mathbf{W}_h = \{\mathbf{u} \in \mathbb{X}_h \mid \langle (\nabla + \mathbf{q}) \cdot \mathbf{u}, r_h\rangle = 0,\ \forall r_h \in Q_h\}$ where Q_h is a suitable scalar finite element space. Due to this relaxation we need to take extra care of the term $a^1(\cdot,\cdot)$, which now also depends on $\mathbf{w}_h$. To obtain an estimate as in (3) we show an inf-sup stability result of the form

$$\inf_{r_h \in Q_h} \sup_{\mathbf{u}_h \in \mathbb{X}_h} \frac{\langle (\nabla + \mathbf{q}) \cdot \mathbf{u}_h, r_h\rangle}{\|\mathbf{u}_h\|_{1,h}\|r_h\|_{Q_h}} \geq c$$

for a constant c independent of the discretization parameters.

References

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