Adaptive Spectral Decomposition for Time-Dependent Inverse Problems

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Abstract

Inverse medium problems involve the reconstruction of a spatially varying medium, $u(x)$, from available observations. Typically, they are formulated as PDE-constrained optimization problems and solved by an inexact Newton-like iteration. Clearly, standard grid-based representations of u are very general but often too expensive due to the resulting high-dimensional search space. Adaptive spectral inversion (ASI) instead expands the unknown medium in a basis of eigenfunctions of a judicious elliptic operator, which depends itself on the current iterate. Rigorous L^2-error estimates of the adaptive spectral (AS) approximation are proved for an arbitrary piecewise constant medium.

Keywords: wave equation, inverse scattering, regularization

1 Inverse scattering problem

Consider the time-dependent wave equation

$$\frac{\partial^2 y}{\partial t^2} - \nabla \cdot (u(x)\nabla y) = f \quad \text{in } \Omega \times (0,T), \qquad (1)$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ and time interval $0 < t < T$, together with appropriate initial and boundary conditions on the boundary Γ of Ω. Here $u(x)$ denotes the (unknown) squared wave speed inside Ω and $f(t,x)$ is a known source.

Given (noisy) observations y_ℓ^{obs} on Γ of solutions of (1) with $f = f_\ell$, $\ell = 1, \dots, N_s$, we seek to determine the medium u. Thus, we formulate the inverse problem as a PDE-constrained optimization problem for the standard L^2-least-squares misfit functional,

$$\mathcal{J}(u) = \frac{1}{2}\sum_{\ell=1}^{N_s}\int_0^T \|y_\ell - y_\ell^{\text{obs}}\|^2_{L^2(\Gamma)}, \qquad (2)$$

where for each ℓ, $y_\ell = y_\ell(u)$ is the solution of (1) with $f = f_\ell$.

2 Adaptive spectral inversion (ASI)

AS decompositions (ASD) have been proposed as low dimensional search spaces during the iterative process of inverse medium problems; see

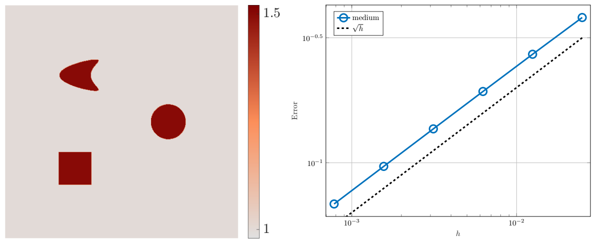

Figure 1: (left) true medium u; (right) the ASD truncation error (8) as a function of h for fixed $\varepsilon = 10^{-8}$ and $K = 3$.

[1, 2] and the references therein. Here we consider an extension of the ASI approach from [2] to the time-dependent wave equation.

Since the inverse medium problem is in general severely ill-posed, a Tikhonov (TV) regularization term is typically added to the misfit (2). Here, instead, we rely on the regularizing effect of the search space selection, which we adapt iteratively as follows: at the m-th iteration we compute u_m by minimizing (2) in a low-dimensional search space Ψ^m. Then, based on u_m and Ψ^m, we build a new search space Ψ^{m+1} for the next iteration.

To construct Ψ^{m+1}, we combine the previous search space Ψ^m with the AS space $\Phi_{K_m} = \text{span}\{\varphi_k\}_{k=1}^{K_m}$ spanned by the first K_m eigenfunctions of the linear elliptic operator [1, 2]

$$L_\varepsilon[u_m]v = -\nabla \cdot (\mu_\varepsilon[u_m]\nabla v), \qquad (3)$$

where

$$\mu_\varepsilon[u_m] = \frac{1}{\sqrt{|\nabla u_m|^2 + \varepsilon^2}}, \quad \varepsilon > 0. \qquad (4)$$

Thus, for each k

$$\begin{aligned} L_\varepsilon[u_m]\varphi_k &= \lambda_k \varphi_k && \text{in } \Omega, \\ \varphi_k &= 0 && \text{on } \Gamma, \end{aligned} \qquad (5)$$

where $(\lambda_k)_{k\geq 1}$ is the nondecreasing sequence of the eigenvalue of $L_\varepsilon[u_m]$, each repeated according to its multiplicity, and $\{\varphi_k\}$ are L^2-orthonormal.

3 Error analysis for ASD

So far, the remarkable ability of the ASD to (approximately) decompose piecewise constant functions has only been supported by numerical evidence. Here we present rigorous L^2-error estimates [3] for AS approximations of piecewise constant functions.

For simplicity, suppose that $u : \Omega \to \mathbb{R}$ is piecewise constant with K inclusions,

$$u(x) = \sum_{k=1}^{K} \alpha_k \chi_{A_k}(x), \quad \alpha_k \neq 0, \tag{6}$$

where χ_{A_k} are the characteristic functions of Lipschitz domains $A_k \subset\subset \Omega$ with connected but mutually disjoint boundaries. Now, let $u_h \in V_h$ be the (standard) interpolant of u in an H^1-conforming, piecewise polynomial $\mathcal{P}^r$-FE space V_h associated with a mesh $\mathcal{T}_h$ of size h, where the family $\{\mathcal{T}_h\}_h$ is regular and quasi-uniform.

Suppose $\{\varphi_k\}_{k=1}^K \subset V_h$ are the (discrete approximate) eigenfunctions obtained by the Galerkin FE formulation of the eigenvalue problem (5) with u_m replaced by u_h. Then the AS projection $\Pi_K^\varepsilon[u_h] : L^2(\Omega) \to \Phi_K$ into the AS space $\Phi_K = \operatorname{span}\{\varphi_k\}_{k=1}^K$ is the standard L^2-projection given by

$$\langle v - \Pi_K^\varepsilon[u_h]v, \varphi \rangle = 0, \qquad \forall \varphi \in \Phi_K. \tag{7}$$

We have the following error estimate [3]:

Theorem 1 *For each* $v \in \operatorname{span}\{\chi_{A_k}\}_{k=1}^K$ *there exists a constant* $C = C(v)$, *such that for every* $\varepsilon, h > 0$ *sufficiently small,*

$$\|v - \Pi_K^\varepsilon[u_h]v\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon + h}. \tag{8}$$

In particular, the above is true for $v = u$.

Remark 2 *The estimate of Theorem 1 holds true in a more general setting [3] where the FE formulation is replaced by a Galerkin formulation in a closed subspace* $\mathcal{V}^\delta \subset H^1$ *and the FE-interpolant* u_h *is replaced by a more general* admissible approximation $u_\delta \in \mathcal{V}^\delta$. *Moreover,* u *also need not be constant near* Γ *and the weight function* (4) *can be replaced by a more general function.*

Consider the piecewise constant medium u shown in Figure 1 (left), which consists of $K = 3$ characteristic functions (obstacles) in $\Omega = (0,1)^2$. To verify the convergence rate in Theorem 1 for $h \to 0$ and fixed $\varepsilon = 10^{-8}$, we compute the approximation error (8) using $\mathcal{P}^1$-FE on a regular, uniform triangular mesh whose vertices lie on an equidistant Cartesian grid with mesh size h. The right frame of Figure 1 corroborates the expected error decay of $\mathcal{O}(\sqrt{h})$.

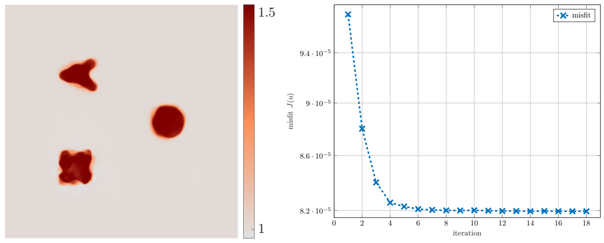

Figure 2: (left) reconstructed medium with 20% noise; (right) misfit referring to (2).

4 Numerical Results

Here we apply the ASI approach, described in Section 2, to solve a time-dependent inverse scattering problem for the (unknown) medium u shown in Figure 1, given the scattered wave data on Γ from 32 evenly distributed sources near the boundary with 20% added noise. The forward problem (1) is discretized in $\Omega = (0,1)^2$ until time $T = 1.5$ using $\mathcal{P}^2$-FE (with mass-lumping) in space and the (standard, explicit, second-order) leapfrog method in time. In Figure 2, we display the reconstructed medium after 18 ASI iterations, when the discrepancy principle is satisfied: Starting from the homogeneous background, the ASI algorithm recovers $u(x)$ both in shape and magnitude quite accurately. Instead of a grid-based discrete FE representation with $250'000$ unknowns, the dimension of the search space in the ASI approach never exceeds $K_{\max} = 100$ during the entire inversion. At each iteration, the first few eigenfunctions φ_k are computed with $\mathcal{P}^1$-FE on the same FE mesh using a cheap Lanczos method for symmetric and positive definite eigenvalue problems.

References

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