Order-preserving non-conforming grid interfaces for boundary-optimized summation-by-parts operators

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Abstract

We present new operators for coupling boundary-optimized summation-by-parts (SBP) finite difference methods with non-equispaced grid points across non-conforming grid interfaces. The coupling utilizes projection operators that move grid functions across the non-conforming interface by projecting them to an intermediate piecewise polynomial space. We demonstrate that the new operators can be used to construct order-preserving interpolation operators for equations with second derivatives in space. The results are corroborated by solving the acoustic wave equation on a grid with a non-conforming interface. In addition, we demonstrate the superior accuracy of the boundary-optimized operators compared to the traditional SBP finite difference operators by solving the Euler vortex problem on a curvilinear grid with a non-conforming interface.

Keywords: high-order finite difference methods, summation-by-parts, non-conforming interfaces

1 Introduction

High-order SBP finite differences have proven to be an efficient and robust method for solving hyperbolic partial differential equations. For linear well-posed problems, the SBP property in combination with stable boundary treatment, such as simultaneous approximation terms (SAT), allows for the construction of provably energy-stable schemes. The SBP property however comes at the cost of a reduction in the order of accuracy of the operator in the boundary region. To alleviate the issue, boundary-optimized SBP operators were derived in [1]. The operators utilize non-equispaced grid points in the boundary region in order to reduce the leading order error. However, the non-equispaced grid points complicate coupling of grids with non-conforming interfaces. Traditionally, non-conforming interfaces are treated by the use of SBP-preserving interpolation operators, first presented in [2]. The interpolation operators transfer grid functions directly from one side of the interface to the other, and are constructed to adhere to a specific grid-spacing ratio across the interface (e.g 1:2). With non-equispaced grid points, no fixed ratio exists. We therefore turn to methods capable of coupling general grids and utilize the method presented in [3], where the grid functions are projected to and from a so-called *glue grid*, i.e., a piecewise polynomial space, in such a way that accuracy and stability is preserved. The interpolation- and glue projection operators in [2, 3] all have order p, leading to a reduction in global convergence rate from $p+2$ to $p+1$ for equations with second derivatives in space discretized using SBP operators of interior order $2p$, and boundary order p, denoted SBP$(2p, p)$. In [4], *order-preserving* interpolation operators were presented. The authors showed that by combining operator pairs of orders $p+1, p$ one can construct an interface coupling such that the reduction in global convergence rate is avoided. Here, we present glue projection operators of orders $p+1, p$ and use these to construct order-preserving couplings of non-conforming grids.

2 Glue projection operators

Consider two finite difference grids $\bar{\Omega}_a$ and $\bar{\Omega}_b$ sharing a common interface. The idea presented in [3] is to construct operators P_{a2g_a}, P_{g_a2a}, where P_{a2g_a} accurately projects grid functions from the interface of $\bar{\Omega}_a$ to a glue grid $\mathcal{G}_a$, and vice versa for P_{g_a2a}. The piecewise polynomial space on $\mathcal{G}_a$ is such that the interval edges coincide with the interface points of $\bar{\Omega}_a$. Once the grid function is projected to the glue grid, going between polynomial spaces is done by standard L^2 projection. That is, for a glue grid $\mathcal{G}_b$ aligned with $\bar{\Omega}_b$ one may construct $P_{g_a2g_b}$ going from $\mathcal{G}_a$ to $\mathcal{G}_b$ by L^2 projection. Given glue projection operators P_{b2g_b} and P_{g_b2b}, the interpolation operator I_{a2b} moving grid functions from $\bar{\Omega}_a$ to $\bar{\Omega}_b$ is formed as $I_{a2b} = P_{g_b2b}P_{g_a2g_b}P_{a2g_a}$. The construction of I_{b2a} is analogous. As previously mentioned, the operators presented in [3] corre-

sponding to SBP$(2p, p)$ operators are of order p. In this work, we construct operator pairs where the order is raised to $p+1$ in one of the projection directions. That is, we construct pairs of operators (P_{a2g_a}, P_{g_a2a}) of orders $(p+1, p)$, and (P_{b2g_b}, P_{g_b2b}) of orders $(p, p+1)$. Following the terminology of [4], this allows us to construct a 'good' I_{a2b} of order $p+1$ and 'bad' I_{b2a} of order p. By exchanging the orders, one may also construct a 'bad' I_{a2b} and a 'good' I_{b2a}. With this set of 'good' and 'bad' operators, the order-preserving interpolation presented in [4] is realized.

3 Numerical results

Consider the following wave equation

$$\begin{aligned} u_{tt} - c_a^2 \Delta u &= 0, & \bar{x} &\in \Omega_a, & t &> 0, \\ v_{tt} - c_b^2 \Delta v &= 0, & \bar{x} &\in \Omega_b, & t &> 0, \\ u - v &= 0, & \bar{x} &\in \Omega_a \cap \Omega_b, & t &> 0, \\ c_a^2 u_x - c_b^2 v_x &= 0, & \bar{x} &\in \Omega_a \cap \Omega_b, & t &> 0, \end{aligned} \tag{1}$$

where $c_a = 1$, $c_b = 1/3$, $\Omega_a = [-10, 0] \times [0, 10]$ and $\Omega_b = [0, 10]^2$. We discretize (1) as in [4] using boundary-optimized SBP operators and our new glue projection operators, with N^2 grid points on $\bar{\Omega}_a$ and $(3N)^2$ grid points on $\bar{\Omega}_b$, such that the number of grid points per wavelength is constant. Convergence is measured against the solution $u = \cos(x + y - \sqrt{2}c_a t) + k_2 \cos(x - y + \sqrt{2}c_a t)$, $v = (1 + k_2)\cos(k_1 x + y - \sqrt{2}c_a t)$, $k_1 = \sqrt{2c_a^2/c_b^2 - 1}$, $k_2 = (c_a^2 - c_b^2 k_1)/(c_a^2 + c_b^2 k_1)$ at $t = 2$ in the SBP H-norm, and the expected order-preserved rate is observed. See Figure 1.

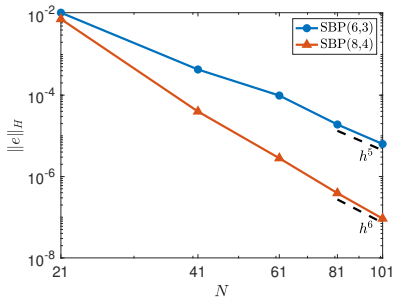
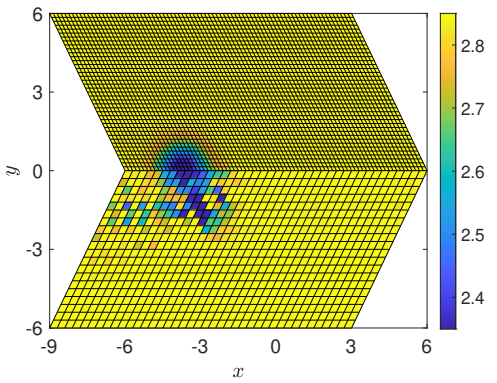
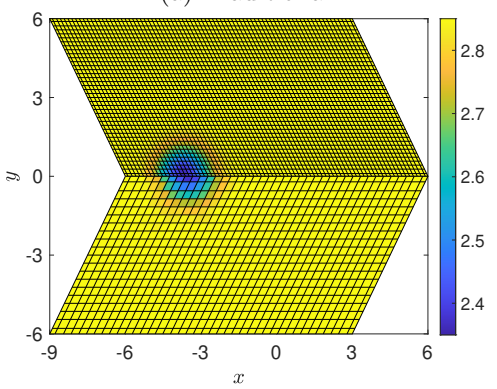


Figure 1: Error plot for the wave equation.

Next, consider the Euler vortex problem on a non-conforming grid with periodic boundaries discretized using SBP$(4, 2)$ operators. See Figure 2 in which the grid is outlined and [1, 2] where the problem is presented in detail. The vortex is initialized in the center of the domain and propagated to the right along the interface. The solution using traditional SBP operators and the interpolation operators in [2] is clearly distorted while similar errors are not visible when using the boundary-optimized operators and the newly developed glue projection operators.



(a) Traditional



(b) Boundary-optimized

Figure 2: Pressure at $t = 8.8$.

References

[1] K. Mattsson, M. Almquist, and E. van der Weide, Boundary optimized diagonal-norm SBP operators, *J. Comput. Phys.* **374** (2018), pp. 1261–1266.

[2] K. Mattsson, and M. Carpenter, Stable and accurate interpolation operators for high-order multiblock finite difference methods, *SIAM J. Sci. Comput.* **4** (2010), pp. 2298–2320.

[3] J. Kozdon, and L. Wilcox, Stable coupling of nonconforming, high-order finite difference methods, *SIAM J. Sci. Comput.* **38** (2016), pp. A923–A952.

[4] M. Almquist, S. Wang, and J. Werpers, Order-preserving interpolation for summation-by-parts operators at nonconforming grid interfaces, *SIAM J. Sci. Comput.* **41** (2019), pp. A1201–A1227.