On the use of phased array data for the Linear Sampling Method in an elastic waveguide

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Abstract

We are interested here in using the Linear Sampling Method (LSM) [1] to image defects in an elastic waveguide [2] by using scattering data for sensors on the surface of the waveguide [3]. In particular, we use phase laws between sensors to emit more energy for each acquisition to improve the quality of the data and so the imaging results. However, the use of these phase laws has an impact on the condition number of the matrices used in the inversion, inducing that close attention must be paid to the acquisition parameters to improve the imaging.

Keywords: Linear Sampling Method, surface data, waveguide, elasticity, time domain

1 Setting of the problem

We consider a two dimensional waveguide $W = \Sigma \times \mathbb{R}$ of transverse section Σ and boundary $\Gamma = \Gamma_0 \cup \Gamma_d$, d being the height of the waveguide and Γ_0 and Γ_d being respectively its lower and upper boundary. This waveguide is made of an isotropic material of density ρ and Lamé constants $(\lambda,\ \mu)$. We want to retrieve the shape of a defect D, the support of which is supposed to be for axis coordinates bigger than $-R$, from data acquired with sources and measurements on the surface Γ. We consider a family of measurement points and sources $(\mathbf{g}(x-x_i)\chi_i(t))_{1\le i\le M}$ located on Γ_d of compact support centered at $x_i = -R - i\delta$, $i = 1, \dots, M$. The displacement $\mathbf{u}$ solves the following problem for a given source $\mathbf{f}$:

$$\begin{cases} \rho\partial_{t^2}\mathbf{u} - \mathrm{div}\sigma(\mathbf{u}) = 0 & \text{in} \quad (W\backslash\overline{D}) \times (0,+\infty), \\ \sigma(\mathbf{u})\nu = \mathbf{f} & \text{on} \quad \Gamma \times (0,+\infty), \\ \mathbf{u} = 0 & \text{on} \quad \partial D \times (0,+\infty), \\ \mathbf{u} = \partial_t\mathbf{u} = 0 & \text{on} \quad (W\backslash\overline{D}) \times \{0\}, \end{cases}$$

where $\sigma(\mathbf{u}) = \lambda(\mathrm{div}\mathbf{u})\, I + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the stress and ν is the exterior normal to W. A common acquisition setup is to use each emitting transducer successively, that is $\mathbf{f}(x,t) = \mathbf{g}(x-x_i)\chi_i(t)$, while all receivers record the field, which has already been used for the LSM [3]. However, this leads in general to poor signal to noise ratios in the data as the transducers emit limited unfocused energy in the waveguide. That is why we consider here to emit simultaneously with all sources but with different delay laws for each acquisition, that is consider sources of expression

$$\mathbf{f}(x,t) = \sum_{i=1}^{M} \mathbf{g}(x-x_i)\chi(t-(M-i)\tau_k),$$

where τ_k, $k = 1, \dots, N$, are delays which must be chosen well to enable proper imaging results. We hereafter briefly describe the LSM in a modal formulation and its extension to surface data, which enables the choice of these delays. We hence consider in the following our problem in the frequency domain, the two problems being linked by the Fourier transform.

2 The Linear Sampling Method: a modal formulation

Decompose $\mathbf{u} = (\mathbf{u}_S, u_3)^T$, $\sigma\mathbf{e}_3 = (\mathbf{t}_S, -t_3)^T$, where the subscripts S and 3 denote the components of a vector along the transverse section and along the axis, respectively. We introduce the mixed variables $\mathbf{X}$, $\mathbf{Y}$ defined by $\mathbf{X} = (\mathbf{t}_S, u_3)^T$, $\mathbf{Y} = (\mathbf{u}_S, t_3)^T$. The guided modes are the solutions with separated variables to $\mathrm{div}\sigma(\mathbf{u}) + \rho\omega^2\mathbf{u} = 0$ in W with boundary condition $\sigma(\mathbf{u})\nu = 0$ on Γ. They are given for $n \in \mathbb{N}$ by

$$\begin{pmatrix} \mathbf{X}_n^{\pm}(x) \\ \mathbf{Y}_n^{\pm}(x) \end{pmatrix} = \begin{pmatrix} \pm\mathcal{X}_n(\mathbf{x}_S) \\ \mathcal{Y}_n(\mathbf{x}_S) \end{pmatrix} e^{\pm i\beta_n x_3},$$

with $(\mathcal{X}_n, \mathcal{Y}_m)_\Sigma = \delta_{mn}$, where $(\cdot,\cdot)_\Sigma$ is a scalar product over $L^2(\Sigma)$ without complex conjugation. For a given frequency ω, β_n is real for only a finite number P of guided modes, which are named propagating modes. The other ones are either inhomogenous or evanescent and are not taken into account here. The assumption is then made that any elastic field, written in the

$(\mathbf{X},\mathbf{Y})$ variables, can be decomposed as follows:

$$\mathbf{X}|_\Sigma=\sum_n(\mathbf{X},\mathcal{Y}_n)_\Sigma\mathcal{X}_n,\quad \mathbf{Y}|_\Sigma=\sum_n(\mathcal{X}_n,\mathbf{Y})_\Sigma\mathcal{Y}_n.$$

The scattered field $\mathbf{u}^s_n$ (and its $\mathbf{Y}$ extension $\mathbf{Y}^s_n$) associated to the incident propagating mode $\mathbf{u}^+_n$ is solution of the following forward problem for a given frequency ω:

$$\begin{cases} \mathrm{div}\sigma(\mathbf{u}^s_n)+\rho\omega^2\mathbf{u}^s_n=0 & \text{in} \quad W\backslash\overline{D}, \\ \sigma(\mathbf{u}^s_n)\nu=0 & \text{on} \quad \Gamma, \\ \mathbf{u}^s_n=-\mathbf{u}^+_n & \text{on} \quad \partial D, \\ (RC), & \end{cases}$$

with (RC) a radiation condition. The data in this case are the components of one reflection block of the scattering matrix $\mathcal{S}$, namely the projections $\mathcal{S}^{+-}_{mn}$ along the $\mathcal{X}_m$ on the section Σ_{-R} of the scattered fields $\mathbf{Y}^s_n$, the number of lines and columns being limited to P. The LSM consists in solving the following system for all sampling points $z=(z_S,z_3)$:

$$\sum_{n=0}^{P-1}\mathcal{U}^{+-}_{mn}h^-_n=\mathrm{e}^{\mathrm{i}\beta_m(R+z_3)}\mathcal{Y}_m(z_S).\mathbf{p},$$

$m=0,\dots,P-1$, where $\mathcal{U}=\mathcal{S}K$, K is the $P\times P$ diagonal matrix the components of which are $e^{i\beta_nR}/2i\beta_n$, $-R$ is the x_3 coordinate of the section Σ_{-R} and $\mathbf{p}$ is a polarisation parameter. If, roughly speaking, a solution $(h^-_0,\dots,h^-_{P-1})$, is found, then $z\in D$ according to a classic result related to the LSM [2].

3 The case of surface sources and measurements

The method shown above needs data within the waveguide, which is not realistic in the context of Non Destructive Evaluation. We hence consider sources and measurements on Γ_d, for which the diffraction problem satisfied by the total field $\mathbf{u}$ is, for a given source $\mathbf{g}$:

$$\begin{cases} \mathrm{div}\sigma(\mathbf{u})+\rho\omega^2\mathbf{u}=0 & \text{in} \quad W\backslash\overline{D}, \\ \sigma(\mathbf{u})\nu=\mathbf{g} & \text{on} \quad \Gamma_d \\ \sigma(\mathbf{u})\nu=0 & \text{on} \quad \Gamma_0, \\ \mathbf{u}=0 & \text{on} \quad \partial D, \\ (RC). & \end{cases} \tag{1}$$

The corresponding scattered field $\mathbf{u}^s$ is $\mathbf{u}-\mathbf{u}^i$, where $\mathbf{u}^i$ solves the same problem (1) as $\mathbf{u}$ in W without the boundary condition $\mathbf{u}=0$ on ∂D. The data are the components of a matrix $\mathcal{M}$ of general term defined by a single component of the scattered fields measured at points $(d,x_j)_{1\le j\le M}$ for all considered sources. The measurement matrix $\mathcal{M}$ is related to the LSM matrix $\mathcal{U}$ by the relationship $\mathcal{M}=-\mathcal{R}\mathcal{U}\mathcal{E}^T$, where $\mathcal{R}$ and $\mathcal{E}$ are some reception and emission matrices. Inverting this system enables to compute $\mathcal{U}$ and then to apply the modal LSM as in section 2.

The most straightforward acquisition setup is then to use individual sensors as sources, that is $\mathbf{f}(x)=\mathbf{g}(x-x_i)\widehat{\chi}(\omega)$, leading to a poor insonification of the region of interest. The corresponding data are denoted $\mathcal{M}$. Another possibility as already mentioned is to use various time delays between all sources for each acquisition, that is

$$\mathbf{f}(x)=\sum_{n=1}^{M}\mathbf{g}(x-x_n)\widehat{\chi}(\omega)\mathrm{e}^{\mathrm{i}(M-n)\tau_k\omega}.$$

The corresponding data are denoted $\tilde{\mathcal{M}}$. The two data are linked by $\tilde{\mathcal{M}}=\mathcal{M}V$, where V is a Vandermonde Matrix of general coefficient $\mathrm{e}^{i(M-n)\tau_k\omega}$. It is then possible to analyze the condition number of V as to ensure the best choice of delays τ_k.

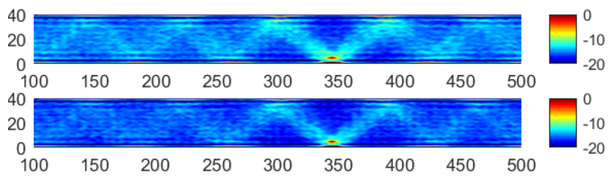

Figure 1: Imaging for simulated data corresponding to successive single sensor emissions (top) and for phased array data (bottom) with 64 sensors (log scale).

References

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