Sheared nanoribbons

Philippe Briet[1,*], **D. Krejčiřík**[2], **Hamza Abdou Soimadou**[1]
[1]Centre de Physique Théorique, Université de Toulon, Toulon, France
[2]Department of Mathematics, Czech Technical University in Prague, Prague, Czechia
*Email: briet@univ-tln.fr

Abstract

The purpose of this note is to study some spectral properties of the Dirichlet Laplacian defined on a two-dimensional infinite band subjected to a "shear". We give geometric conditions leading to a Hardy inequality and the absence of a discrete eigenvalue. The second part is devoted to the discussion of the presence of discrete spectrum. Apart from a few details, the bulk of proof can be found in [1].

Keywords: Quantum waveguide, sheared band, Hardy inequality.

1 Introduction

Recent work has shown that certain deformations of a straight waveguide have a repulsive effect, i.e. the absence of a discrete spectrum of the corresponding Dirichlet Laplacian; see for example BHK, BH, Kre. We want to study this fact for a two-dimensional quantum waveguide subjected to shear. To this end we introduce the following model. Let $f:\ \mathbb{R}\to\mathbb{R}$ such that
(**h**) *the derivative* $f'\in L^\infty_{loc}(\mathbb{R})$ *and* $f'(s)\to\beta$ *as* $|s|\to+\infty$, $\beta\in\mathbb{R}\cup\{\pm\infty\}$.
If $\beta\in\mathbb{R}$ the deviation is denoted by $\varepsilon:=f'(s)-\beta$. Let $d>0$. Consider the domain in $\mathbb{R}^2$ (see Figure 1):

$$\Omega=\Omega_f=\{(x,y)\in\mathbb{R}^2;f(x)<y<f(x)+d\}$$

We are focusing on the spectral analysis of the "Dirichlet Laplacian " denoted by $-\Delta_D$ in $\mathrm{L}^2(\Omega)$ i.e. the self-adjoint operator in $\mathrm{L}^2(\Omega)$ defined from the quadratic form

$$\mathcal{Q}_D[\psi]=\int_\Omega|\nabla\psi(x,y)|^2dxdy,\quad \psi\in\mathrm{H}^1_0(\Omega).$$

For finite β it is convenient to use an appropriate change of variables:

$$(s,t)\in\Omega_0\longrightarrow\mathcal{L}(s,t)=(s,f(s)+t)\in\Omega$$

Denote by H_f the operator obtained in the curvilinear coordinates (s,t), it is associated to the following quadratic form :

$$q[\varphi]=\|(\partial_s-f'\partial_t)\varphi\|^2+\|\partial_t\varphi\|^2;\varphi\in D(q)\quad (1)$$

By direct calculation we see that q is closed on $D(q)=H^1_0(\Omega_0)$. Let us give the location of the essential spectrum of the operator H_f which is a necessary step for our purpose. Let $E_1(\beta)=\left(1+\beta^2\right)E_1$, where $E_1=\left(\frac{\pi}{d}\right)^2$ is the first transverse mode: $-\partial_t^2\chi(t)=E_1\chi(t)$,

Theorem 1 *Suppose (***h***) holds. Then,*
i) if $\beta\in\mathbb{R}$*. Then,* $\sigma_{ess}(H_f)=[E_1(\beta),+\infty)$
ii) if $f'\to\pm\infty$*. Then* $\sigma_{ess}(H)=\emptyset$*.*

The proof of the Theorem 1 can be found in [1]. Note that for $\beta=\pm\infty$ the spectrum of the operator H_f is purely discrete so from now we only consider the finite β case.

2 Hardy inequalities

Theorem 2 (repulsive shearing) *Suppose (***h***) holds,* ε *a nonzero function,* $\beta\in\mathbb{R}$*, and* $\beta\varepsilon\geq 0$*. Then there exists* $c>0$ *s.t.*

$$-\Delta_D-E_1(\beta)\geq\frac{c}{1+s^2}\qquad (2)$$

holds in the quadratic form sense in $L^2(\Omega)$*.*

Let us give few remarks. The last theorem implies the non existence of discrete eigenvalue for H_f. If $\varepsilon=0$, a limiting argument show that (2) cannot be true see [2]. Finally, note also that the presence of positive term in the r.h.s of (2) shows that the result is stable by adding a small perturbation to H_f of order 0.

The proof of the Theorem is given in [1]. The key point comes from the so called "the ground state decomposition", it is the following identity which is valid for every finite β and $\varepsilon\in L^\infty_{loc}(\mathbb{R})$. Let $\psi\in C_0^\infty(\Omega_0)$, then

$$\begin{aligned}q[\psi]-E_1(\beta)\|\psi\|^2&=\|\partial_s\psi-\varepsilon\partial_t\psi-\beta\chi\partial_t(\chi^{-1}\psi)\|^2\\&\quad+\|\chi\partial_t(\chi^{-1}\psi)\|^2\\&\quad+\int_{\Omega_0}\beta\varepsilon\left(E_1(\beta)+(\frac{\chi'}{\chi})^2\right)|\psi|^2\quad (3)\end{aligned}$$

Then by (3), since the r.h.s. is positive if $\beta\varepsilon\geq 0$ then the associated operator H_f has no spectrum below $E_1(\beta)$ even for $\varepsilon=0$.

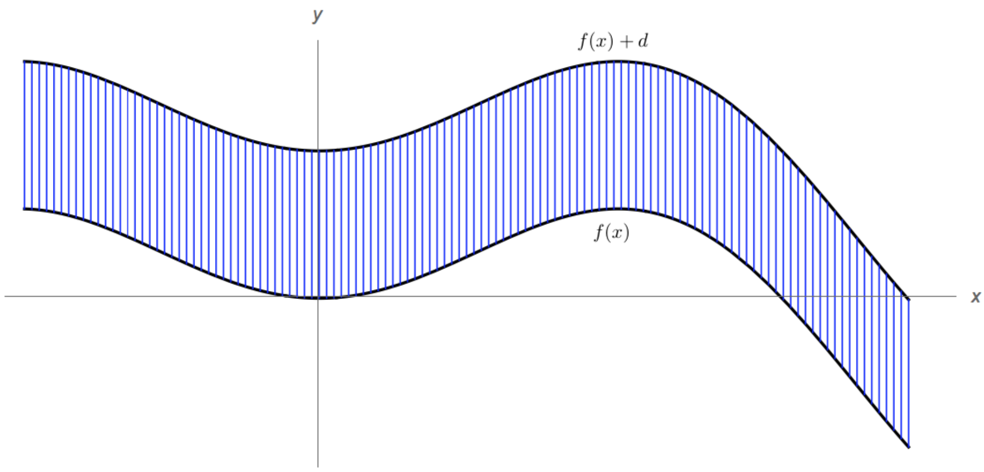

Figure 1: sheared nanoribbon

3 Discrete spectrum

Theorem 3 (Attractive shearing) *Suppose that ε is a nonzero function, $\varepsilon^2 + 2\beta\varepsilon \in L^1(\mathbb{R})$ and either one of the following conditions is satisfied,*

$$\int_{\mathbb{R}} (\varepsilon^2 + 2\beta\varepsilon) < 0 \tag{4}$$

$$\varepsilon \in W^1_{loc}(\mathbb{R}), \varepsilon \neq -2\beta \text{ and } \int_{\mathbb{R}} (\varepsilon^2 + 2\beta\varepsilon) = 0 \tag{5}$$

Then $\sigma_d(H_f) \neq \emptyset$.

The criterion (4) of existence of discrete eigenvalues has been used in [5] for a different model. In fact here we prove more i.e. the discrete eigenvalues persist even when (4) is saturated (see (5)). This last result is more delicate to obtain and we need here an additional condition on the regularity of the deviation. Of course this result is consistent with Theorem 2 since for repulsive shearing assumptions (4) and (5) are clearly not satisfied.

4 Large coupling

We close this work, by considering a particular case namely f is s.t. $f'(x) = \beta + \alpha\varepsilon(x)$, $\beta > 0$, $\alpha < 0$, ε is a bounded positive function with support $[0, 1]$. Then we get:

Theorem 4 *Suppose that there exist $0 < c_1 \leq \varepsilon(x) \leq c_2$, $x \in [0, 1]$. Then for $\alpha < 0$ and large $\sigma_d(H_f) = \emptyset$.*

References

[1] Abdou-Soimadou H, Briet P., Krejčiřík D. Spectral analysis of sheared nanoribbons. *Z. Angew. Math. Phys.* **70**, (2) (2019), 18 pp.

[2] Briet P., Hammedi H., Krejčiřík D., Hardy inequality in a globally twisted waveguide, *Lett. Math. Phys.*, 2015. **105** pp. 939–958.

[3] Briet P., Hammedi H., Twisted waveguide with a Neumann window. Functional Analysis and Operator Theory for Quantum Physics : The Pavel Exner Anniversary Volume, European Mathematical Society, pp.161-175, 2017, EMS Series of Congress Reports, Eur. Math. Soc., Zürich, 2017.

[4] D. Krejčiřík, twisting versus bending in quantum waveguides, Analysis on Graphs and Applications (Cambridge 2007), Proc. Sympos. PureMath., *Amer. Math. Soc.*, Providence, RI, **77** (2008), pp.617–636.

[5] Exner P., Kovarik H., Spectrum of the Schrödinger operator in a periodically twisted tube , *Lett. Math. Phys.*, 2005. **73** pp. 183–192.