One-Way methods for wave propagation in complex flows

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Abstract

We propose a numerical factorization of the propagation operator in a privileged direction for linearized Navier-Stokes equations. From the latter, One-Way models such as the True Amplitude formalism and the Bremmer series can be derived.

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1 Introduction

The One-Way (OW) approach is an effective tool to compute wave propagation in a direction of interest. For example, such a method could be used to study the laminar-turbulent transition zone in boundary layers. However, the construction of this type of method in this context remains a challenge due to complex operators involved.

Towne & Colonius [1] have overcome this difficulty for slowly varying flows by proposing a purely numerical OW method based on a non-reflecting boundary condition applied to the Euler and Navier-Stokes equations.

We extend these works by proposing a new factorization of the propagation operator which allows a sorting of the modes according to their direction and accessing to a refraction/reflection operator. We can then derive OW models with broader validity, such as True Amplitude OW (TAOW) or Bremmer series.

2 Linearized Navier-Stokes equations

We are interested in computing with a low numerical cost the time-harmonic wave propagation in a complex flow, and considering a preferred direction: the x-axis in this presentation. To do so, we start with a 2D linearized model of the Navier-Stokes equations around a mean flow where the unknown fluctuation vector is $\widetilde{\mathbf{q}} = (\widetilde{\nu}, \widetilde{u}, \widetilde{v}, \widetilde{p})^T$ with $\widetilde{\nu}$, $(\widetilde{u}, \widetilde{v})$ and $\widetilde{p}$ being the specific volume, velocity and pressure perturbations, respectively. Using standard hypothesis in wave propagation, the second derivatives in x are neglected. We next process as in [1] by isolating the term $\frac{\partial}{\partial x}$ and by performing a first discretization in the transverse direction y (a compact high-order finite difference scheme in our case). So, we get the following matrix ODE in x-variable:

$$\mathbf{A}\frac{d\widetilde{\mathbf{q}}}{dx} = \underbrace{(i\omega\mathbf{I} - \mathbf{B}_y\mathbf{D}_y - \mathbf{B}_{yy}\mathbf{D}_{yy} - \mathbf{C})}_{\mathbf{B}}\widetilde{\mathbf{q}} \tag{1}$$

with $\mathbf{D}_y$ and $\mathbf{D}_{yy}$ the first and second order discrete derivative operators in y-variable. All the matrices are of size $(4N_y) \times (4N_y)$ with N_y the number of discretization points in the transverse direction y. The matrix $\mathbf{A}$ is assumed to be diagonalizable, *i.e.* $\mathbf{A} = \mathbf{T}\,\widetilde{\mathbf{A}}\,\mathbf{T}^{-1}$ and invertible. If it contains singularities, they can be treated by extraction.

The equation (1) can be rewritten in terms of characteristic variables $\boldsymbol{\phi} = \mathbf{T}^{-1}\,\widetilde{\mathbf{q}}$:

$$\frac{d\boldsymbol{\phi}}{dx} = \mathbf{M}(x)\boldsymbol{\phi} \tag{2}$$

where the propagation operator $\mathbf{M}$ is defined by:

$$\mathbf{M} = \widetilde{\mathbf{A}}^{-1}\mathbf{T}^{-1}\mathbf{B}\mathbf{T} - \mathbf{T}^{-1}\frac{d\mathbf{T}}{dx}.$$

3 New factorization of M

The construction of OW methods to solve (2) requires identifying the left and right-going modes contained in the propagation operator. A natural way is to perform a diagonalization of $\mathbf{M}$ and to analyze the behavior of the eigenvalues to find the different sets of modes (Briggs' criterion). Unfortunately, this approach is generally costly in terms of computational resources, especially because $\mathbf{M}$ is x-dependent. To bypass this problem, we propose to construct a new decomposition of $\mathbf{M}$ based on the concept of high-order non-reflecting boundary conditions. The starting point is this remark: a strict diagonalization of $\mathbf{M}$ is not needed to realize a OW decoupling.

More precisely, by using this type of matrices

$$\begin{aligned}\mathbf{Z}^{r,N} &:= \prod_{j=0}^{N-1}(\mathbf{M}-i\beta_-^j\mathbf{I})^{-1}(\mathbf{M}-i\beta_+^j\mathbf{I})\\ \mathbf{Z}^{l,N} &:= \prod_{j=0}^{N-1}(\mathbf{M}-i\beta_+^j\mathbf{I})^{-1}(\mathbf{M}-i\beta_-^j\mathbf{I})\end{aligned}$$

with $(\beta_+^j,\beta_-^j)_{j=0,\dots,N-1}$ a set of parameters, the following theorem can be proven.

Theorem 1 *Let* **VDU** *be a diagonalization of* **M** *where* **D** *contains the* $N_\pm$ *eigenvalues* $i\alpha_\pm$ *corresponding to the* $\pm x$ *direction. The parameters* $(\beta_\pm^j)_{j\in\mathbb{N}}$ *satisfy the condition (C):*

$$\lim_{N\to+\infty} R^N(\alpha_+) = 0 \text{ and } \lim_{N\to+\infty} R^N(\alpha_-) = +\infty$$

with $R^N(\alpha) := \prod_{j=0}^{N-1}\left|\frac{\alpha-\beta_+^j}{\alpha-\beta_-^j}\right|$.
We introduce the matrix

$$\widetilde{\mathbf{U}}^N := \begin{pmatrix} \mathbf{I}_{++} & \left(\mathbf{Z}^{l,N}_{++}\right)^{-1}\mathbf{Z}^{l,N}_{+-} \\ \left(\mathbf{Z}^{r,N}_{--}\right)^{-1}\mathbf{Z}^{r,N}_{-+} & \mathbf{I}_{--}\end{pmatrix}$$

where the subscripts denote the number of rows and columns (N_+ or N_-) of each block.

The following decoupling result holds: for $\widetilde{\mathbf{D}}^N := \widetilde{\mathbf{U}}^N\,\mathbf{M}\,(\widetilde{\mathbf{U}}^N)^{-1}$, *let* $\widetilde{\mathbf{D}} := \lim_{N\to+\infty}\widetilde{\mathbf{D}}^N$,

$$\widetilde{\mathbf{D}} = \begin{pmatrix}\mathbf{U}_{++}^{-1}\,\mathbf{D}_{++}\mathbf{U}_{++} & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_{--}^{-1}\,\mathbf{D}_{--}\mathbf{U}_{--}\end{pmatrix}$$

This theorem implies that by choosing a finite number of parameters $\beta_\pm^j$, we can construct

$$\widetilde{\mathbf{D}}^N = \widetilde{\mathbf{U}}^N\,\mathbf{M}\,(\widetilde{\mathbf{U}}^N)^{-1} = \begin{bmatrix}\widetilde{\mathbf{D}}^N_{++} & \boldsymbol{\varepsilon}\\ \boldsymbol{\varepsilon} & \widetilde{\mathbf{D}}^N_{--}\end{bmatrix} \quad (3)$$

where $\widetilde{\mathbf{D}}^N_{++}$ and $\widetilde{\mathbf{D}}^N_{--}$ are two block matrices containing the information about the right-going and left-going eigenvalues of $\mathbf{M}$, respectively and $\boldsymbol{\varepsilon}$ represents the residuals of the approximations, which will be neglected.

This choice is guided by (C). It is easy to find such kind of family, for example, $\beta_\pm^{j-1} = \pm(j+ij)$ for $j\in\mathbb{N}^*$ but the convergence of the method can be greatly improved by using parameters nearer to the spectrum of $\mathbf{M}$. In practice, we use some *a priori* information on the spectrum of $\mathbf{M}$ such as the branch position of evanescent modes, the range of the convective modes branch *etc.*, which are deduced from analytical results for simple mean flows [1] or from a numerical local stability analysis (see example below).

By using the factorization (3), the equation (2) can be written in terms of OW variables $\boldsymbol{\psi} = \widetilde{\mathbf{U}}^N\boldsymbol{\phi}$:

$$\frac{d\boldsymbol{\psi}}{dx} = \widetilde{\mathbf{D}}^N\boldsymbol{\psi} + \mathbf{W}^N\boldsymbol{\psi}$$

with $\mathbf{W}^N := -\widetilde{\mathbf{U}}^N\frac{d\widetilde{\mathbf{V}}^N}{dx}$ the refraction/reflection matrix. Finally, we construct several OW propagation models: for example, right-going OW equations are derived as follows

$$\frac{d\boldsymbol{\psi}_+}{dx} = \widetilde{\mathbf{D}}^N_{++}\boldsymbol{\psi}_+ + \delta\mathbf{W}^N_{++}\boldsymbol{\psi}_+ \text{ and } \boldsymbol{\psi}_- = \mathbf{0}$$

with $\delta = 0$ for the standard OW and $\delta = 1$ for the TAOW. In the case of Bremmer series, we need to solve right and left OW equations iteratively until convergence.

4 Numerical example

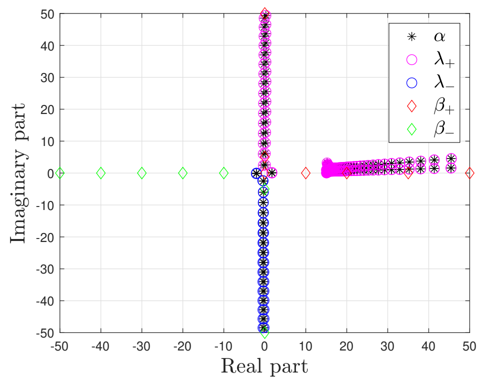

Figure 1: Convergence on the spectrum: $N = 7$

We consider a duct of height h and a mean flow verifying $(\overline{\nu},\overline{p})$ constant and the velocity is:

$$\overline{u}(y) = 4V_{max}\frac{y}{h}\left(1-\frac{y}{h}\right) \text{ and } \overline{v} = 0$$

where V_{max} is the maximum axial velocity.

In Fig. 1, α corresponds to the exact operator spectrum, λ_+ and λ_- to the spectra of the operators $\widetilde{\mathbf{D}}^N_{++}$ and $\widetilde{\mathbf{D}}^N_{--}$, respectively. We see a good restitution of the spectrum from few $\beta_\pm$ coefficients. During the congress, we will present numerical results obtained by some OW methods based on this new factorization.

References

[1] A. Towne and T. Colonius, One-Way spatial integration of hyperbolic equations, *Journal of Computational Physics*, **300** (2015), pp. 844—861.