Asymptotic analysis for sound-hard acoustic scattering by two closely-situated spheres

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Abstract

We consider acoustic binding of particles resulting from radiation forces created through multiple scattering. This problem has potential for developing methods for assembling novel metamaterials. A key consideration in acoustic binding is when two or more particles are closely situated to one another and form a *cluster*. For that case, the near-field scattering by the particles becomes important. Here, we study multiple scattering by two closely-situated sound-hard spheres. Using boundary integral equation (BIE) methods, we find that a close evaluation problem arises leading to a nearly singular system of BIEs governing the surface fields. An asymptotic analysis of the problem reveals that this nearly singular behavior will lead to large error in the numerical solution unless it is explicitly addressed.

Keywords: Boundary integral methods, asymptotics, nearly-singular integrals.

1 Problem Description

We consider the plane wave $u^{\text{inc}} = e^{ikz}$ scattered by two sound-hard spheres (denoted D_i, with boundary B_i, $i = 1, 2$). The total field $u = u^{\text{sca}} + u^{\text{inc}}$ satisfies

$$\begin{aligned} &\Delta u + k^2 u = 0, \text{ in } E := \mathbb{R}^3 \setminus (\overline{D}_1 \cup \overline{D}_2) \\ &\partial_n u = 0, \text{ on } \partial E := B_1 \cup B_2, \end{aligned} \quad (1)$$

and u^{sca} satisfies the Sommerfeld radiation condition. Upon solution of (1), we compute the acoustic radiation forces,

$$\begin{aligned} \mathbf{F}_i = -\int_{B_i} \Big\{ \Big[\frac{1}{2}\kappa_0 \langle p_1^2 \rangle - \frac{1}{2}\rho_0 \langle v_1^2 \rangle \Big] \hat{n} \\ + \rho_0 \langle (\hat{n} \cdot \vec{v}_1)\vec{v}_1 \Big\} \mathrm{d}S, \quad i = 1, 2, \end{aligned} \quad (2)$$

where $\vec{v}_i = \nabla u_i$, $p_i = \mathrm{i}\rho_0 \omega u_i$, with ρ_0 is the density of the spheres, ω is the angular frequency, $\hat{n}$ the unit outward normal of each sphere, and u_i is the surface field on each sphere: $u_i = u|_{B_i}$, $i = 1, 2$. It has been shown recently that this force can be significant when the size of the spheres are comparable to the wavelength and lead to so-called acoustic binding of particles [1]. An important case in acoustic binding is when the spheres are situated closely to one another. To study this problem, let $\mathbf{x}_1$ and $\mathbf{x}_2$ denote the centers of the two spheres both with radius a (with $ka = O(1)$), and consider $|\mathbf{x}_1 - \mathbf{x}_2| = 2a + \varepsilon$ in the asymptotic limit, $\varepsilon \to 0^+$ (see Fig. 1). This scattering problem is challenging due to strong near-fields interactions.

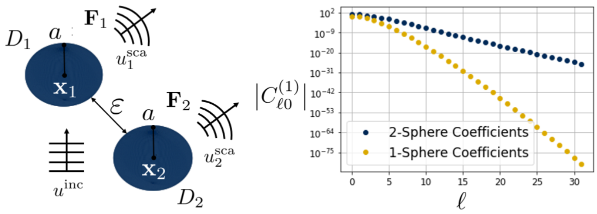

Figure 1: (Left) Sketch of the problem and notations. (Right) Plot of the expansion coefficients $(C_{\ell 0}^{(1)})_{\ell = [\![0,32]\!]}$ in (5) for $k = 7.33$, $a = 0.16$, $\mathbf{x}_1 = (0,0,0)$, $\mathbf{x}_2 = (0,0,2a+1)$ ($\varepsilon = 1$). The blue curve represents obtained $(C_{\ell 0}^{(1)})_\ell$ when approximating (4) using Gaussian Product Quadrature rule (GPQ), the yellow curve refers to $(C_{\ell 0}^{(1)})_\ell$ obtained analytically when there is only D_1.

2 Boundary integral equations

Note that (2) only requires the surface fields. For this reason, boundary integral equation methods are natural for studying this problem. Additionally, since the scatterers are spheres, we use a Galerkin projection method to study the governing boundary integral equations (BIEs) that we discuss below. We write

$$u^{\text{sca}}(x) = \sum_{i=1}^{2} \mathcal{S}_i[\partial_n u_i^{\text{inc}}] + \mathcal{D}_i[u_i^{\text{sca}}], \; x \in E, \quad (3)$$

where $\mathcal{D}_i$ and $\mathcal{S}_i$, $i = 1, 2$ are the double- and single-layer potentials, respectively, for each of the two spheres indexed by i. In (3), $\mathcal{S}_i$ is applied to the normal derivative of the known incident field on B_i, which we denote by $\partial_n u_i^{\text{inc}}$, and $\mathcal{D}_i$ is applied to the unknown scattered field on B_i, which we denote by u_i^{sca}. Upon projecting (3) onto B_1 and B_2, we obtain the following system of BIEs,

$$\begin{bmatrix} \frac{1}{2}-\mathcal{D}_{11} & -\mathcal{D}_{12} \\ -\mathcal{D}_{21} & \frac{1}{2}-\mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} u_1^{\text{sca}} \\ u_2^{\text{sca}} \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ \mathcal{S}_{21} & \mathcal{S}_{22} \end{bmatrix} \begin{bmatrix} \partial_n u_1^{\text{inc}} \\ \partial_n u_2^{\text{inc}} \end{bmatrix}, \tag{4}$$

where $\mathcal{D}_{ij}$ and $\mathcal{S}_{ij}$ are the double- and single-layer potentials for sphere i evaluated on sphere j, respectively.

3 Galerkin method

To solve (4), we substitute

$$u_i^{\text{sca}} = \sum_{\ell m} C_{\ell m}^{(i)} h_\ell^{(1)}(kr) Y_\ell^m(\theta,\varphi),\ i=1,2, \tag{5}$$

with $h_\ell^{(1)}$ denoting the spherical Hankel functions of the first kind, and $Y_\ell^m(\theta,\varphi)$ the spherical harmonics. Similarly, one can write a spherical harmonic expansion of the layer potential kernels. Using the orthogonality of spherical harmonics, we derive a linear system for the expansion coefficients $C_{\ell m}^{(i)}$, $i=1,2$. We truncate this system at some fixed order ℓ^*, solve that system, and use the corresponding truncation of (5) to approximate the solution.

We solve this truncated system for the expansion coefficients with $\varepsilon = 1$. The expansion coefficients $C_{\ell 0}^{(1)}$ for B_1 as a function of order ℓ are shown as a blue dotted curve in the right plot in Fig. 1. For comparison, we plot the expansion coefficients for a single sphere as a yellow dotted curve (analytic result). The decay of $C_{\ell m}^{(1)}$ with ℓ is substantially slower than that for a single sphere. Thus, one needs to consider large values of ℓ^* in comparison to the single sphere problem to achieve a comparable accuracy.

The slower decay of expansion coefficients is due to the off-diagonal blocks in (4). Those off-diagonal blocks model the coupling of scattered fields between the two spheres. In particular they involve the evaluation of the double-layer potential from one sphere on the other. In the limit $\varepsilon \to 0^+$, we find that these off-diagonal blocks involve a close evaluation of the double-layer potential thereby yielding nearly singular behavior. The results shown in Fig. 1 indicate that even when the spheres are not too close, the coupling effects are strong and affect the accuracy of the numerical approximation.

4 Asymptotic Analysis

Without addressing the close evaluation problem explicitly, one will need large computational resources to maintain accuracy [2, 3]. To identify the main cause and address this problem, we propose an asymptotic analysis of the off-diagonal blocks in (4). Below, we identify the main mechanism resulting in the nearly singular behavior of those off-diagonal blocks.

At close evaluation distance $r = a(1+\epsilon)$, by expanding about $\epsilon = 0$ one can show that

$$\begin{aligned} h_\ell^{(1)}(ka(1+\epsilon)) = \Bigg[\left(1-\epsilon^2\frac{(ka)^2}{2}\right) \\ +\epsilon(1-\epsilon)\ell + \epsilon^2\frac{\ell(\ell+1)}{2} \Bigg] h_\ell^{(1)}(ka) \\ -\epsilon(1+\epsilon)(ka)h_{\ell+1}^{(1)}(ka) + O(\epsilon^3). \end{aligned} \tag{6}$$

Using (6), one can show that the close evaluation of the double-layer potential involves two operators: the spherical Laplacian Δ_{S^2} which satisfies $\Delta_{S^2} Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$, and

$$L_{3/2}[u] = \frac{1}{4\pi\sqrt{2}} \int_{S^2} \frac{u(\hat{s}') - u(\hat{s})}{(1-\hat{s}\cdot\hat{s}')^{3/2}} \mathrm{d}\hat{s}', \tag{7}$$

which satisfies $L_{3/2} Y_{\ell m} = -\ell Y_{\ell m}$ [4]. These operators account for the slow decay of expansion coefficients at close separation distance (causing large errors). In the spirit of [3], one can derive asymptotic approximations of the operators above, allowing to design an adapted quadrature rule for the off-diagonal terms that addresses the close evaluation problem. Extensions to the scattering by multiple spheres and comparison of computed acoustic radiation forces with experiments will be considered.

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