On corner matrices for high order DDMs

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Abstract

Our problem concerns the construction of domain decomposition methods (DDM) for the model Helmholtz equation with high order transmission conditions (2nd order TC in our case) and cross-points. A compatibility condition is formulated for cross-points matrices so that the DDM is proved to be convergent under general conditions. The proof is based on a new global energy formulated on the skeleton of the DDM decomposition. The equivalence of the new energy with the H^1 nom on the skeleton of the DDM decomposition is shown. The extension at any order is briefly discussed.

Keywords: DDM, second-order TC, cross-points.

1 Introduction

We consider a general family of domain decomposition iterative processes:

- Initialize $u_i^0 \in H^1(\Omega_i)$ for all i.
- For $p \in \mathbb{N}$, compute $u_i^{p+1} \in H^1(\Omega_i)$ for all i

$$\left\{ \begin{array}{rcl} \left(\Delta + \omega^2\right) u_i^{p+1} &=& f \text{ in } \Omega_i, \forall i, \\ \left(\partial_{\mathbf{n}} - \mathbf{i}\omega T\right) u_\Sigma^{p+1} &=& -\left(\Pi \partial_{\mathbf{n}} + \mathbf{i}\omega T \Pi\right) u_\Sigma^p, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega) u_\Gamma^{p+1} &=& 0 \text{ on } \Gamma, \end{array} \right. \tag{1}$$

where the second equation denotes a possibly global transmission condition on the skeleton Σ of the DDM. The operator Π denotes the natural exchange operator on Σ.
We report hereafter on recent advances [5] (to compare with [1–3]) where the operator T comes from a second order approximation of transparent condition on a flat boundary in 2D. That is we desire that the transmission equation T on Σ in (1) models

$$\begin{array}{l} \left(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}_i\mathbf{t}_i}\right) \partial_{\mathbf{n}_i} u_{ij}^{p+1} - \mathbf{i}\omega u_{ij}^{p+1} \\ = -\left(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}_j\mathbf{t}_j}\right) \partial_{\mathbf{n}_j} u_{ji}^{p} - \mathbf{i}\omega u_{ji}^{p} \text{ on } \Sigma_{ij}, \end{array} \tag{2}$$

where $\Sigma_{ij} = \Sigma_{ji}$ is the part of Σ in between Ω_i and Ω_j. Such a requirement is algorithmically natural since second order transmission operators are already implemented in [4]. One notices that we have selected a second order operator which has some positivity property since the principal symbol is $1 - \frac{1}{2\omega^2}\partial_{\mathbf{tt}} \geq 0$ which is formally non negative.
We propose to focus on combinations of Neumann traces and Dirichlet traces under the form

$$\partial_{\boldsymbol{\tau}_{ij}} \varphi_{ij}(\mathbf{x}_r) + \sum_{(k\ell)\in\mathcal{E}_r} \alpha_{ij,k\ell}^r \varphi_{k\ell}(\mathbf{x}_r) = 0, \ \forall (ij) \in \mathcal{E}_r, \tag{3}$$

where $\mathcal{E}_r$ denotes the set of edges around the node/vertex $\mathbf{x}_r$, $\boldsymbol{\tau}_{ij}$ denotes a tangential derivation and φ_{ij} denotes with simpler notations a linear rescaling of $\partial_{\mathbf{n}_i} u_{ij}$. All quantities (3) which concern Neumann traces and Dirichlet traces around the same node/vertex $\mathbf{x}_r$ are gathered in two vectors, one for the Neumann traces and one for the Dirichlet traces. One obtains

$$\partial_{\boldsymbol{\tau}} \varphi_r + A^r \varphi_r = 0, \tag{4}$$

with the matrix $A^r \in \mathcal{M}_{2d_r}(\mathbb{C})$ contains the unknowns coefficients $(\alpha_{ij,k\ell}^r)_{ij,k\ell}$.

2 Definition of T and other considerations

We model/introduce the above considerations by defining the bilinear form $a(\cdot,\cdot)$ on the skeleton: for all $\varphi, \psi \in H^1_{\mathrm{br}}(\Sigma)$

$$a(\varphi,\psi) = \sum_{i,j} \int_{\Sigma_{ij}} \left(\varphi_{ij}\overline{\psi_{ij}} + \frac{1}{2\omega^2} \partial_{\mathbf{t}_i}\varphi_{ij} \overline{\partial_{\mathbf{t}_i}\psi_{ij}} \right) \mathrm{d}\sigma + \frac{1}{2\omega^2} \sum_{r=1}^{N_X} (A^r \varphi_r, \psi_r)_{\mathbb{C}^{2d_r}}. \tag{5}$$

We make the assumption that A^r has pure imaginary coefficients and that it is skew-hermitian $A^r = \mathbf{i}H^r$ where $H^r = (H^r)^{\mathsf{T}} \in \mathcal{M}_{2d_r}(\mathbb{R})$. Since a is coercive (in particular because the transmission operator is symmetric non negative), one can define

$$\left\{ \begin{array}{l} \text{Find } \varphi \in H^1_{\mathrm{br}}(\Sigma) \text{ such that} \\ a(\varphi,\psi) = (v,\psi)_{L^2(\Sigma)}, \quad \forall \psi \in H^1_{\mathrm{br}}(\Sigma). \end{array} \right. \tag{6}$$

This problem is well posed, there exists a unique solution $\varphi \in H^1_{\mathrm{br}}(\Sigma)$, which denotes the skeleton endowed with the natural broken H^1 norm.

Definition 1 *Let $T\colon L^2(\Sigma) \to L^2(\Sigma)$ be the operator such that $Tv = \varphi$, where φ is the solution to the problem* (6) *for $v \in L^2(\Sigma)$.*

The symmetric part of T admits a spectral decomposition. Let $(u_n)_{n\in\mathbb{N}} \subset H^1_{\mathrm{br}}(\Sigma)$ be the Hilbertian basis such that $\left(\frac{T+T^*}{2}\right) u_n = \lambda_n u_n$, $(u_n, u_m)_{L^2(\Sigma)} = \delta_{nm}$ and $\overline{\mathrm{span}\{u_n\}_{n\in\mathbb{N}}}^{L^2(\Sigma)} = L^2(\Sigma)$. The eigenvalues satisfy $1 \geq \lambda_n \geq \lambda_{n+1} > 0$ and λ_n converges towards zero as n goes to infinity. This leads to the definition of the operator $\left(\frac{T+T^*}{2}\right)^{-1} : L^2(\Sigma) \to L^2(\Sigma)$ such that $\left(\frac{T+T^*}{2}\right)^{-1} u_n = \frac{1}{\lambda_n} u_n$ and the space $H^1_T(\Sigma) := \left\{u \in L^2(\Sigma),\ |||u||| < \infty\right\}$, endowed with the norm $|||\cdot|||$ defined as

$$|||u||| := \left(\sum_{n\geq 0} \frac{\left|(u, u_n)_{L^2(\Sigma)}\right|^2}{\lambda_n}\right)^{1/2}, \quad \forall u \in L^2(\Sigma).$$

The hermitian scalar product in $H^1_T(\Sigma)$ is denoted $\langle u, v\rangle := \sum_{n\geq 0} \frac{1}{\lambda_n} (u, u_n)_{L^2(\Sigma)} \overline{(v, u_n)}_{L^2(\Sigma)} = \left(\left(\frac{T+T^*}{2}\right)^{-1} u, v\right)_{L^2(\Sigma)}$, $\forall u, v \in H^1_T(\Sigma)$. As a consequence, $H^1_T(\Sigma)$ is a Hilbert space. One of our main mathematical result is that $H^1_T(\Sigma) = H^1_{\mathrm{br}}(\Sigma)$ with equivalence of norms.

Definition 2 *We say that operator T is compatible if $\Pi T \Pi = T^*$.*

Under the compatibility assumption, DDM (1) can be rewritten as

$$\left\{\begin{array}{rcl} \left(\Delta + \omega^2\right) u_i^{p+1} & = & f \text{ for all } i, \\ \left(\partial_{\mathbf{n}} - \mathbf{i}\omega T\right) u_\Sigma^{p+1} & = & -\Pi \left(\partial_{\mathbf{n}} + \mathbf{i}\omega T^*\right) u_\Sigma^p, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega) u_\Gamma^{p+1} & = & 0 \text{ on } \Gamma. \end{array}\right. \tag{7}$$

Define $E^p := \left|\left|\left|\left(\partial_{\mathbf{n}} - \mathbf{i}\omega T\right) u_\Sigma^p\right|\right|\right|^2$.

Lemma 3 *Assume that T verifies the compatibility condition from Definition 2, and that at each stage of the algorithm* (7) *$u^p \in \oplus_i H^1(\Omega_i)$ and $\partial_{\mathbf{n}} u_\Sigma^p \in H^1_{\mathrm{br}}(\Sigma)$. Then, algorithm* (7) *is stable (and finally convergent) in the sense that its energy is decreasing: $E^{p+1} = E^p - 4\omega^2 \left\|u^p\right\|^2_{L^2(\Gamma)}$.*

3 Extension to higher order transmission conditions

A natural question for our methodology is to model transmission conditions at higher order. Let us consider a plane wave

$$u(\mathbf{x}) = e^{\mathbf{i}\omega(\cos\theta x + \sin\theta y)}.$$

One considers the exact outgoing condition on a flat boundary $\{x = 0\}$ such that $\partial_{\mathbf{n}} = \partial_x$: $\partial_{\mathbf{n}} u - \mathbf{i}\omega \cos\theta u = 0$. For $|\sin\theta| < 1$, one has the convergent expansion $\frac{1}{\cos\theta} = (1 - \sin^2\theta)^{-\frac{1}{2}}$ $= 1 + \frac{\sin^2\theta}{2} + \frac{3\sin^4\theta}{8} + \frac{5\sin^6\theta}{16} + \frac{35\sin^8\theta}{128} + \dots$ All coefficients are positive. Then a truncation at any order yields a transparent boundary condition of the corresponding order. For example, an expansion at order 6 writes $\left(1 + \frac{\sin^2\theta}{2} + \frac{3\sin^4\theta}{8} + \frac{5\sin^6\theta}{16}\right) \partial_{\mathbf{n}} u - \mathbf{i}\omega u = 0$. One has formally $\mathbf{i}\omega \sin\theta = \partial_y = \partial_{\mathbf{t}}$. One obtains the artificial condition

$$\left(1 - \frac{\partial^2_{\mathbf{tt}}}{2\omega^2} + \frac{3\partial^4_{\mathbf{tttt}}}{8\omega^4} - \frac{5\partial^6_{\mathbf{tttttt}}}{16\omega^6}\right) \partial_{\mathbf{n}} u - \mathbf{i}\omega u = 0.$$

By construction, the operator $A = 1 - \frac{\partial^2_{\mathbf{tt}}}{2\omega^2} + \frac{3\partial^4_{\mathbf{tttt}}}{8\omega^4} - \frac{5\partial^6_{\mathbf{tttttt}}}{16\omega^6}$ is formally symmetric non negative. For such an operator one can defines generalized Neumann traces ($\partial^r_{\mathbf{t}}\varphi$ for $r = 3, 4, 5$) and Dirichlet traces ($\partial^r_{\mathbf{t}}\varphi$ for $r = 0, 1, 2$), then it is possible to define a new bilinear form (5) with generalized corner matrices. The properties of this new bilinear a and of the corresponding new transmission operator is left for further studies.

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