High frequency asymptotic expansions for sound hard multiple scattering problems

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Abstract

We derive asymptotic expansions of multiple scattering total fields for the exterior high frequency sound hard scattering problem relating to a finite collection of disjoint smooth compact strictly convex obstacles.

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1 Introduction

In the last decade, the high frequency sound soft plane-wave scattering problem has found an extensive interest in the context of smooth compact and strictly convex scatterers (see [6, 8, 9] and the references therein). The methods developed for single scattering problems in some of those references are extended to tackle multiple scattering configurations [2–4, 7, 10].

The sound hard single scattering problem, on the other hand, has only been recently investigated [1]. This, in part, is due to the complicated form of the asymptotic expansion for the Neumann condition when compared to its Dirichlet counter part [12] (see [11] for an alternative approach for the Dirichlet problem). These ansatzs are respectively given by

$$\eta(x,k) \sim \sum_{\substack{p,q,r\geq 0\\ \ell\leq -1}} k^{-\frac{1+2p+3q+r+\ell}{3}+(\ell+1)_-} \, b_{p,q,r,\ell}(x)\,(\Psi^{r,\ell})^{(p)}(k^{\frac{1}{3}}Z(x)) \tag{1}$$

for the Neumann case, and

$$\eta(x,k) \sim \sum_{p,q\geq 0} k^{\frac{2-2p-3q}{3}}\, b_{p,q}(x)\,\Psi^{(p)}(k^{\frac{1}{3}}Z(x))$$

for the Dirichlet problem. Here we extend the expansion (1) to multiple scattering problems.

2 Multiple scattering problem

We consider the two dimensional sound hard scattering problem in the exterior of finitely many smooth compact strictly convex obstacles $K = \bigcup\{K_\sigma : \sigma \in \mathcal{J}\}$ illuminated by a plane wave incidence $u^{\text{inc}}(x) = e^{ik\,\alpha\cdot x}$ ($k > 0$, $|\alpha| = 1$). The unknown scattered field u satisfies the sound hard scattering problem [5]

$$\begin{cases} (\Delta + k^2)u = 0 \text{ in } \mathbb{R}^2\backslash K,\\ \partial_\nu u = -\partial_\nu u^{\text{inc}} \text{ on } \partial K,\\ \lim\limits_{r\to\infty} \sqrt{r}\big(\partial_r u - iku\big) = 0,\ r = |x|, \end{cases}$$

(ν is the exterior unit normal to ∂K), and can be formally reconstructed as an appropriate superposition of multiple scattering iterations [2, 10].

In this context, it suffices to consider the iterative solution of the following problems

$$\begin{cases} (\Delta + k^2)u_0 = 0 \text{ in } \mathbb{R}^2 \setminus K_0,\\ \partial_\nu u_0 = -\partial_\nu u^{\text{inc}} \text{ on } \partial K_0,\\ \lim\limits_{r\to\infty} \sqrt{r}\big(\partial_r u_0 - iku_0\big) = 0, \end{cases} \tag{2}$$

and for $m \geq 1$

$$\begin{cases} (\Delta + k^2)u_m = 0 \text{ in } \mathbb{R}^2 \setminus K_m,\\ \partial_\nu u_m = -\partial_\nu u_{m-1} \text{ on } \partial K_m,\\ \lim\limits_{r\to\infty} \sqrt{r}\big(\partial_r u_m - iku_m\big) = 0, \end{cases} \tag{3}$$

where $\{K_m\}_{m\geq 0} \subset \{K_\sigma : \sigma \in \mathcal{J}\}$ is any sequence with $K_{m+1} \neq K_m$. Solutions of problems (2)-(3), in turn, can be recovered through the double layer representations

$$u_m(x,k) = \int_{\partial K_m} \frac{\partial G(x,y)}{\partial \nu(y)}\, \eta_m(y,k)\, ds(y)$$

(G is the outgoing Green's function for Helmholtz equation) provided the *multiple scattering total fields*

$$\eta_m = \begin{cases} u_0 + u^{\text{inc}}, & m = 0,\\ u_m + u_{m-1}, & m \geq 1, \end{cases} \quad \text{on } \partial K_m,$$

are known.

3 Asymptotic expansions of multiple scattering total fields

As in the sound soft case [2, 10], the geometrical conditions

$$\{x + t\alpha : x \in K_0, t \geq 0\} \cap K_1 = \varnothing,$$
$$K_{m+1} \cap \overline{co}(K_m \cup K_{m+2}) = \varnothing,$$

($\overline{co}$ is the closed convex hull) allow for the extraction of the phases of multiple scattering total fields in the form

$$\eta_m(x,k) = e^{ik\,\varphi(x)}\,\eta_m^{\text{slow}}(x,k).$$

The design and analysis of numerical methods for the efficient solution integral equations related to the multiple scattering total fields η_m is based on the asymptotic expansions of the envelopes η^{slow} [1, 7–10]. In this connection, we have:

Theorem 1 *For all $m \geq 0$, $\eta_m^{\text{slow}}(x,k)$ belongs to the Hörmander class $S^0_{\frac{2}{3},\frac{1}{3}}(\partial K_m \times (0,\infty))$ and admits an asymptotic expansion*

$$\eta_m^{\text{slow}}(x,k) \sim \sum_{\substack{p,q,r\geq 0\\ \ell\leq -1}} a_{m,p,q,r,\ell}(x,k)$$

with

$$a_{m,p,q,r,\ell}(x,k) = k^{-\frac{1+2p+3q+r+\ell}{3}+(\ell+1)_-} \\ b_{m,p,q,r,\ell}(x)\,(\Psi^{r,\ell})^{(p)}(k^{\frac{1}{3}}Z_m(x))$$

where $b_{m,p,q,r,\ell}$ is a complex valued smooth function, Z_m is a real valued smooth function which is positive on the illuminated region ∂K_m^{IL}, negative on the shadow region ∂K_m^{SR}, and vanishes precisely to first order on the shadow boundary ∂K_m^{SB}, and $\Psi^{r,\ell}$ is a complex valued smooth function which admits an asymptotic expansion

$$\Psi^{r,\ell}(\tau) \sim \sum_{j=0}^{\infty} a_{r,\ell,j}\tau^{1+\ell-2r-3j}, \quad \text{as } \tau \to \infty,$$

and rapidly decreases in the sense of Schwartz as $\tau \to -\infty$.

Remark 2 *We refer to [2, 6, 9, 10] for definitions of the illuminated region ∂K_m^{IL}, the shadow region ∂K_m^{SR}, and the shadow boundary ∂K_m^{SB}.*

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