Three types of quasi-Trefftz functions for the 3D convected Helmholtz equation: construction and theoretical approximation properties

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Abstract

Trefftz methods are numerical methods for the approximation of solutions to boundary and/or initial value problems. They are Galerkin methods with particular test and trial functions, which solve locally the governing partial differential equation (PDE). This property is called the Trefftz property. Quasi-Trefftz methods were introduced to leverage the advantages of Trefftz methods for problems governed by variable coefficient PDEs, by relaxing the Trefftz property into a so-called quasi-Trefftz property: test and trial functions are not exact solutions but rather local approximate solutions to the governing PDE. In this presentation we will tackle the question of the definition, construction and approximation properties of three families of quasi-Trefftz functions for 3D scalar PDEs: two based on generalizations on plane wave solutions, and one polynomial.

Keywords: quasi-Trefftz bases, best approximation properties

1 Introduction

In the general context of wave propagation, the application of Trefftz methods in their standard form to problems of propagation through inhomogeneous media is limited since exact solutions are not known for most variable-coefficient equations. Quasi-Trefftz methods, relying on approximate solutions - called quasi-Trefftz functions - rather than exact solutions to the governing equation, were introduced to extend Trefftz methods to problems governed by variable-coefficient equations. They were first introduced in [3] under the name of Generalized Plane Wave (GPW) methods for 2D problems governed by the Helmholtz equation and their approximation properties were studied in [1]. The original idea behind the GPW concept was to retain the oscillating behavior of a plane wave (PW) while allowing for some extra degrees of freedom to be adapted to the varying PDE coefficient, and this is where their name came from. Initially this was performed via the introduction of Higher Order Terms (HOT) in the phase of a PW as follows:

$$\begin{cases} \varphi(x) = \exp(i\widetilde{\kappa}\mathbf{k}\cdot x + \text{HOT}) \\ \left[-\Delta - \kappa^2\epsilon(x)\right]\varphi(x) \approx 0, \end{cases} \tag{1}$$

instead of:

$$\begin{cases} \phi(x) = \exp i\kappa\mathbf{k}\cdot x \\ \left[-\Delta - \kappa^2\right]\phi = 0, \end{cases} \quad \forall \text{ unit vector } \mathbf{k}\in\mathbb{R}^3,$$

where κ is the wavenumber of the PW while $\widetilde{\kappa}$ can be interpreted as the local wavenumber of the GPW. Moreover the quasi-Trefftz property (being an approximated solution to the governing PDE) was defined as a local property thanks to Taylor expansion: given a desired order q and a point x_C, a quasi-Trefftz function is a function φ satisfying: $\forall x\in\mathbb{R}^2$ near x_C,

$$\left[-\Delta - \kappa^2\epsilon(x)\right]\varphi(x) = O(\|x - x_C\|^q).$$

This presentation, based on [5], focuses on three types of quasi-Trefftz functions for a set of second order PDEs in 3D including the convected Helmholtz equation.

2 Three families of quasi-Trefftz functions

Inspired by classical PWs $\exp\Lambda\cdot\Big(x - x_C\Big)$, we will focus on three different families of quasi-Trefftz functions:

- phase-based GPWs, following the original ansatz proposed in [3] via the introduction of higher order terms in the phase of a PW, see (1),
- amplitude-based GPWs, following the ansatz proposed in [2] via the introduction of higher order terms in the amplitude of a PW,
- purely polynomial quasi-Trefftz functions, which so far we have only used for time-dependent wave propagation in [4].

Each of these is uniquely defined by a polynomial, either the phase polynomial, the amplitude polynomial, or the function itself in the polynomial case. Constructing such functions is then equivalent to constructing these polynomials, while imposing the quasi-Trefftz property yields a system whose unknowns are these polynomial's coefficients. In each case a careful study of the corresponding system leads to the derivation of an algorithm to construct quasi-Trefftz function relying only on explicit formulas. Hence, on the one hand the existence of quasi-Trefftz functions is guaranteed for each of the three families, and on the other hand the computational cost for their construction is limited.

We will describe these algorithms, emphasizing the common structure that relies on the second order derivatives in the governing PDE.

3 Local approximation properties of quasi-Trefftz spaces

Both Trefftz and quasi-Trefftz spaces depend on the governing PDE. When comparing approximation properties of Trefftz spaces with standard polynomial spaces, there are two important differences: (1) to reach a given (high) order of approximation, Trefftz spaces require less degrees of freedom than polynomial spaces, (2) Trefftz spaces have high order approximation properties (as polynomial spaces) but only to approximate solutions to the governing PDE. The situation is the same for Quasi-Trefftz spaces, and their approximation properties can be expressed as follows.

Theorem 1 *Consider a desired order of approximation $n \in \mathbb{N}$, a point $x_C \in \mathbb{R}^3$, and a second order partial differential operator $\mathcal{L}_c$ (see [5] for precise hypothesis). Consider three quasi-Trefftz spaces associated to partial differential operator $\mathcal{L}_c$, defined as the spaces spanned by each of the three following sets:*

- *the set of amplitude-based GPWs,*
- *the set of phase-based GPWs,*
- *the set of polynomial functions,*

each of them constructed with $q = \max(n-1,1)$ and $p = (n+1)^2$ functions. Then there exist a choice of functions such that any of these three spaces, denoted $\mathbb{V}_h^G$, satisfies the following approximation property:

$$\forall u \in \mathcal{C}^{\max(2,n)}(\Omega) \text{ satisfying } \mathcal{L}_c u = 0, \exists u_a \in \mathbb{V}_h^G, \\ \exists C \in \mathbb{R} \text{ s. t. } \forall x \in \Omega, \\ |u(x) - u_a(x)| \leq C\|x - x_C\|^{n+1}, \tag{2}$$

The constant C here depends on the domain Ω, on the desired order n, on the PDE solution u to be approximated, as well as on the PDE coefficients c.

We will highlight the most important steps of the theorem's proof, emphasizing the crucial role played by free parameters introduced in the construction algorithms to (i) construct a family of linearly independent quasi-Trefftz functions, and (ii) leverage the explicit formulas from the algorithms.

References

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