# Identification of a local perturbation in unknown periodic layers

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**Abstract**

We revisit the differential sampling method introduced in [1] for the identification of a local perturbation in unknown periodic layers. We provide a theoretical justification of the method that avoids assuming that the local perturbation is also periodic. Our theoretical framework uses functional spaces with continuous dependence with respect to the Floquet-Bloch variable. The corner stone of the analysis is the justification of the Generalized Linear Sampling Method (GLSM) in this setting, which is the main topic presented here.

***Keywords:*** inverse problem, Periodic layers, Floquet-Bloch Transform, domain reconstruction

## 1 Introduction

We consider in this work the inverse scattering problem for the reconstruction of a local perturbation in unknown periodic layers from near field measurements. This considered problem has connections with many practical applications, such as non-destructive testing of photonic structures, antenna arrays... It has motivated many research works over the recent years [1,2]. The presence of the perturbation does not allow us to reduce the problem to one-period cell, and this is what makes our problem more challenging since we are obliged to treat this problem in an unbounded band.

The GLSM Method was applied in similar setting of problem in [1] by adding a technical hypothesis on the distribution of the defect, they assumed that the defect is itself distributed periodically with a longer period, which allows them to reduce the problem to a larger and bounded domain. Then, in this work, and inspired by [2,3], we propose to get rid of this assumption and to work on the whole unbounded domain.

## 2 Setting of the direct problem

Let $U_0$ be the upper half-space $\mathbb{R} \times \mathbb{R}_+$. We denote by $D^p$ the periodic unbounded domain included in $\Omega^R := \mathbb{R} \times [0, R]$ delimited by $\Gamma^0 := \mathbb{R} \times \{0\}$ and $\Gamma^R := \mathbb{R} \times \{R\}$, with $R \geq R_0 > 0$ as shown in Figure 1. We consider $D := D^p \cup \tilde{D}$ where $\tilde{D}$ is a bounded domain included in $\Omega_0^R := [0, 2\pi] \times [0, R]$. We assume that the complement of $D$ is connected.

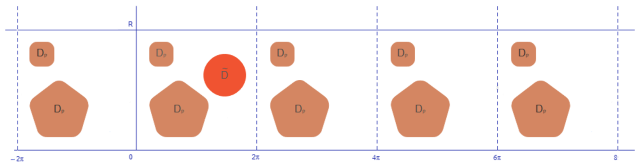

Figure 1: Sketch of the domain

Let $n \in L^\infty(U_0)$ be the refraction index verifying $n = n_p$ outside $\tilde{D}$, where $n_p \in L^\infty(U_0)$ with positive imaginary part, $2\pi$-periodic with respect to the first component $x_1$ such that $n_p = 1$ outside $D^p$. Consider an incident field $v \in L^2(D)$, the direct problem is to find the scattered field $w \in H^2_{loc}(U_0)$ verifying

$$(\mathcal{P}) \begin{cases} \Delta w + k^2 n w = k^2(1-n)v & \text{in } U_0 \\ w = 0 & \text{on } \Gamma^0 \end{cases}$$

and the angular spectrum representation

$$w(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i x_1 \cdot \xi + i\sqrt{k^2 - |\xi|^2}(x_2 - R)} \widehat{w}(\xi, R) d\xi \quad (1)$$

for $x_2 > R$ as a radiation condition, where $\widehat{w}$ is the Fourier transform of the trace of $w$ on $\Gamma^R$. For $s \in \mathbb{R}$, we denote by $H^s_\xi(\Omega_0^R)$ the Sobolev space of the $\xi$-quasi-periodic functions [1], and defining for $\phi \in C_0^\infty(\overline{\Omega^R})$ the Horizontal Bloch-Floquet transform as the following for $\xi \in I = [0, 1]$

$$\mathcal{J}_{\mathbb{R}} \phi(\xi, x_1, x_2) = \sum_{j \in \mathbb{Z}} \phi(x_1 + 2\pi j, x_2) e^{-i 2\pi \xi \cdot j}.$$

which is an isomorphism between $H^s(\Omega^R)$ and $L^2(I, H^s_\xi(\Omega_0^R))$. We also define

$$\tilde{H}^s(\Omega^R) := \left\{ u \in H^s(\Omega^R) | \mathcal{J}_{\mathbb{R}} u \in C^0_\sharp(I, H^s_\xi(\Omega_0^R)) \right\}$$

and similarly for $H^s(\Gamma^R)$. For the sake of studying the inverse problem, for fixed $\xi_0 \in I$, we consider $v_{\xi_0} \in L^2_{\xi_0}(D_0^p)$ and we rewrite the solution of $(\mathcal{P})$ seeking $w_{\xi_0} \in L^2_{loc}(U_0)$ solution of

$$\Delta w_{\xi_0} + k^2 n w_{\xi_0} = k^2(1-n) v_{\xi_0} \text{ in } U_0,$$

which is decomposed as $w_{\xi_0} := w^p_{\xi_0} + \tilde{w}_{\xi_0}$, with $w^p_{\xi_0} \in H^1_{\xi_0}(\Omega^R_0)$ verifying

$$\Delta w^p_{\xi_0} + k^2 n_p w^p_{\xi_0} = k^2(1-n_p)v_{\xi_0} \text{ in } \Omega^R_0$$

and $\tilde{w}_{\xi_0} \in \tilde{H}^1(\Omega^R)$ satisfying

$$\Delta \tilde{w}_{\xi_0} + k^2 n \tilde{w}_{\xi_0} = k^2(n_p - n)(v_{\xi_0} + w^p_{\xi_0}) \text{ in } \Omega^R$$

where both $w^p_{\xi_0}$ and $\tilde{w}_{\xi_0}$ satisfy homogeneous Dirichlet boundary conditions on $\Gamma^0$ and some appropriate radiation conditions (for instance, condition (2) for $\tilde{w}_{\xi_0}$).

## 3 Setting of the inverse problem

Our inverse problem consists in reconstructing $D$ having the measurements $u^s(x,y)$ for all $x$, $y \in \Gamma^R$, where $u^s(.,\cdot)$ is the scattered field solution of $(\mathcal{P})$ with $v = \Phi(\cdot, y)$ the fundamental solution for the Dirichlet half space problem. We notice that for defining this solution, we need the imaginary part of $n_p$ to be positive (at least in a subdomain of $\Omega^R_0 \cap D^p$). We introduce the near field operator $N : \tilde{L}^2(\Gamma_R) \to \tilde{L}^2(\Gamma_R)$

$$Ng = \int_{\Gamma^R} u^s(x,y)g(y)ds(y)$$

We shall explain how the positivity assumption on $n_p$ allows us to define this operator. In fact the imaging procedure does not require all of $N$ but only the Floquet-Bloch transform of $N$ evaluated in few well chosen Floquet-Bloch variables. Let $\xi_0 \in I$ be fixed and define $u^s_{\xi_0}(\cdot, y) = w_{\xi_0}$ for $v_{\xi_0} = \overline{\Phi_{\xi_0}(y,\cdot)}$ where $\Phi_{\xi_0}(x,y) = \mathcal{J}_{\mathbb{R}}\Phi(\cdot,y)(\xi_0,x)$. Using the decomposition of $w_{\xi_0} = w^p_{\xi_0} + \tilde{w}_{\xi_0}$ we decompose $u^s_{\xi_0}(\cdot,y) = u^{s,p}_{\xi_0}(\cdot,y) + \tilde{u}^s_{\xi_0}(\cdot,y)$ and define two operators form $L^2_{\xi_0}(\Gamma^R_0)$ into itself

$$N^p_{\xi_0} g_{\xi_0} = \int_{\Gamma^0_R} u^{s,p}_{\xi_0}(x,y) g_{\xi_0}(y)ds(y)$$
$$\tilde{N}_{\xi_0} g_{\xi_0} = \int_{\Gamma^0_R} \mathcal{J}_{\mathbb{R}}(\tilde{u}^s_{\xi_0}(\cdot,y))(\xi_0,x) g_{\xi_0}(y)ds(y)$$

These operators can be formally constructed using the operator $N$ since one can prove that

$$Ng = \int_I \int_{\Gamma^0_R} u^s_\xi(\cdot,y)\mathcal{J}_{\mathbb{R}} g(\xi,y)ds(y)$$

Using these operators we define the norm

$$I_{\xi_0}(g_{\xi_0}) = |(N^p_{\xi_0} g_{\xi_0}, g_{\xi_0})| + |(\tilde{N}_{\xi_0} g_{\xi_0}, g_{\xi_0})|$$

We now explain how one can use the GLSM to identify $D$ with either the operator $N$ or its Floquet-Bloch transform. The simplest assumptions for which the following theorems hold are when $\tilde{D} \cap D^p = \emptyset$, $(1-n_p)$ and $(n_p - n)$ have fixed sign and definite real parts and negative imaginary parts on $D^p$ and $\tilde{D}$ respectively. We introduce functional $J_\alpha(\phi,\cdot) : \tilde{L}^2(\Gamma^R) \longrightarrow \mathbb{R}$

$$J_\alpha(\phi; g) := \alpha I(g) + \|Ng - \phi\|^2,$$

where $I(g) := \sup_{\xi_0 \in I} I_{\xi_0}(\mathcal{J}_{\mathbb{R}} g(\xi_0,\cdot))$. We denote by $j_\alpha(\phi) = \inf_g J_\alpha(\phi; g)$.

**Theorem 1** *Let $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$ consider $z \in \Omega^R$, and let $g^\alpha \in \tilde{L}^2(\Gamma^R)$ such that*

$$J_\alpha(\Phi(\cdot,z), g^\alpha(z)) \le j_\alpha(\Phi(\cdot,z)) + c(\alpha)$$

*then $z \in D \iff \lim_{\alpha\to 0} I(g^\alpha(z)) < \infty$.*

The second theorem allows to reconstruct $D \cap \Omega^R_0$ using only one Floquet-Bloch mode by minimizing the functional $J^\alpha_{\xi_0} : \tilde{L}^2_{\xi_0}(\Gamma^R_0) \longrightarrow \mathbb{R}$

$$J^\alpha_{\xi_0}(\phi; g_{\xi_0}) = \alpha I_{\xi_0}(g_{\xi_0}) + \|(N^p_{\xi_0} + \tilde{N}_{\xi_0})g_{\xi_0} - \phi\|^2,$$

We denote by $j^\alpha_{\xi_0}(\phi) = \inf_{g_{\xi_0}} J^\alpha_{\xi_0}(\phi; g_{\xi_0})$.

**Theorem 2** *Let $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$ consider $z \in \Omega^R$, and let $g^\alpha_{\xi_0} \in L^2_{\xi_0}(\Gamma^R_0)$ such that*

$$J_\alpha(\Phi_{\xi_0}(\cdot,z), g^\alpha_{\xi_0}(z)) \le j_\alpha(\Phi_{\xi_0}(\cdot,z)) + c(\alpha)$$

*then $z \in D \cap \Omega^R_0 \iff \lim_{\alpha\to 0} I_{\xi_0}(g^\alpha_{\xi_0}(z)) < \infty$.*

Based on Theorem 2 applied to integer multiples of $\xi_0$, we shall explain how one can design an indicator function that allows to directly reconstruct $\tilde{D}$ using a similar differential indicator function as in [1].

## References


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