Linearly implicit energy consistent time discretisation for nonlinear wave equations

Guillaume Castera[1,*], Juliette Chabassier[1]
[1]MAKUTU team, LMAP (UMR 5142, UPPA–Inria BSO)
*Email: guillaume.castera@inria.fr

Abstract

Nonlinear phenomena can occur in vibrating structures, as for instance piano strings, due to large deformations or nonlinear constitutive state laws. Integrating the nonlinear models in space and time can be done accurately in many ways, but preserving an energy identity at the discrete level is an efficient way to adress numerical stability when coupling with other systems (as in the case of the piano). Gradient based integrators achieve this purpose at the cost of solving a nonlinear system at every time step. New formulations called Invariant Energy Quadratization (IEQ) and Scalar Auxiliary Variable (SAV) [2–4] only require the inversion of a linear system while still preserving a discrete energy identity. This work presents a convergence analysis of an interleaved time integrator based on IEQ and θ-scheme as well as numerical illustrations.

Keywords: nonlinear wave equation, energy identity, linearly implicit scheme

1 Introduction

Let $x \in \Omega = [0, L]$ and $t \in [0, T]$, a system of wave equations is considered under the form :

$$M\ddot{q} + (R_1\dot{q} - \partial_x(R_2\partial_x\dot{q})) - \partial_x (K\partial_x q + Bq + \nabla\mathcal{U}(\partial_x q)) + Cq + {}^tB\partial_x q = S(x,t) \quad (1)$$

where $q(x, t) \in \mathbb{R}^N$, M, R_1, R_2, K, B, C are matrices with physical coefficients, $\mathcal{U} : \mathbb{R}^N \mapsto \mathbb{R}^+$ is a nonlinear application and S is the source term. Boundary and initial conditions are given by $q(0, t) = q(L, t) = q(x, 0) = 0$. The piano string enters this general framework, see [1]. Any regular solution to (1) satisfies an energy identity that states

$$\frac{d\mathcal{E}}{dt} = \int_0^L S\cdot\dot{q} - \int_0^L R_1\dot{q}\cdot\dot{q} - \int_0^L R_2\partial_x\dot{q}\cdot\partial_x\dot{q} \quad \text{(2a)}$$

$$\mathcal{E} = \frac{1}{2}\int_0^L M\dot{q}\cdot\dot{q} + \int_0^L \mathcal{U}(\partial_x q) + \frac{1}{2}\int_0^L \begin{pmatrix} C & {}^tB \\ B & K\end{pmatrix}\begin{pmatrix} q \\ \partial_x q\end{pmatrix}\cdot\begin{pmatrix} q \\ \partial_x q\end{pmatrix} \quad \text{(2b)}$$

2 Energy compliant formulation of a nonlinear system of wave equations

Following the Invariant Energy Quadratization technique [2–4] an auxiliary variable is introduced as $\zeta(x,t) = \sqrt{2\mathcal{U}(\partial_x q(x,t)) + c}$, where c is chosen so that the square root is real. Hence the term $\nabla\mathcal{U}(\partial_x q)$ in (1) is equal to $\zeta\,\mathcal{H}(\partial_x q)$ where

$$\mathcal{H}(p) = \frac{\nabla\mathcal{U}(p)}{\sqrt{2\mathcal{U}(p) + c}} \quad (3)$$

so (1) becomes

$$\begin{cases} M\ddot{q} + (R_1\dot{q} - \partial_x(R_2\partial_x\dot{q})) - \partial_x (K\partial_x q + Bq \\ \qquad + \zeta\,\mathcal{H}(\partial_x q)) + Cq + {}^tB\partial_x q = S(x,t) \\ \dot{\zeta} = \mathcal{H}(\partial_x q)\cdot\partial_x\dot{q} \end{cases} \quad (4)$$

and this new system satisfies the previous energy identity (2a) with

$$\mathcal{E}_c = \frac{1}{2}\int_0^L M\dot{q}\cdot\dot{q} + \frac{1}{2}\int_0^L \zeta^2 + \frac{1}{2}\int_0^L \begin{pmatrix} C & {}^tB \\ B & K\end{pmatrix}\begin{pmatrix} q \\ \partial_x q\end{pmatrix}\cdot\begin{pmatrix} q \\ \partial_x q\end{pmatrix} \quad (5)$$

3 Space and time approximations

After performing a variational formulation of (4) with $q(t) \in (H^1(\Omega))^N$ and $\zeta(t) \in L^2(\Omega)$ and restricting to adequate finite-elements approximation spaces, a semi-discrete system is obtained

$$\begin{cases} M_h\ddot{Q}_h + R_h\dot{Q}_h + K_hQ_h + {}^t\mathbb{H}(Q_h)Z_h = S_h \\ A_h\dot{Z}_h = \mathbb{H}(Q_h)\cdot\dot{Q}_h \end{cases} \quad (6)$$

where M_h, A_h, R_h and K_h are usual FEM matrices, S_h the source vector, and $\mathbb{H}(Q_h)$ is a matrix whose values depend on Q_h.

Following [4], an interleaved time scheme is proposed:

$$\begin{cases} M_h\,[Q_h]^n_{\Delta t^2} + R_h\frac{Q_h^{n+1}-Q_h^{n-1}}{2\Delta t} + K_h\,\{Q_h\}^n_\theta + \\ \qquad {}^t\mathbb{H}(Q_h^n)\frac{Z_h^{n+1/2}+Z_h^{n-1/2}}{2} = S_h^n \\ A_h\frac{Z_h^{n+1/2}-Z_h^{n-1/2}}{\Delta t} = \mathbb{H}(Q_h^n)\,\frac{Q_h^{n+1}-Q_h^{n-1}}{2\Delta t} \end{cases} \quad (7)$$

where $[Q_h]^n_{\Delta t^2} = (Q_h^{n+1} - 2Q_h^n + Q_h^{n-1})/\Delta t^2$ and $\{Q_h\}^n_\theta = \theta Q_h^{n+1} + (1-2\theta)Q_h^n + \theta Q_h^{n-1}$.

3.1 Energy consistency

This space/time discretisation satisfies a discrete equivalent of (2a)

$$\frac{\mathcal{E}_h^{n+1/2} - \mathcal{E}_h^{n-1/2}}{\Delta t} = S_h^n \cdot \frac{Q_h^{n+1} - Q_h^{n-1}}{2\Delta t} - \left\| \frac{Q_h^{n+1} - Q_h^{n-1}}{2\Delta t} \right\|^2_{R_h}$$

$$\mathcal{E}_h^{n+1/2} = \frac{1}{2}\left\| \frac{Q_h^{n+1} - Q_h^n}{\Delta t} \right\|^2_{\widetilde{M}_h} + \frac{1}{2}\left\| \frac{Q_h^{n+1} + Q_h^n}{2} \right\|^2_{K_h} + \frac{1}{2}\left\| Z_h^{n+1/2} \right\|^2_{A_h}$$

where the modified mass matrix writes $\widetilde{M}_h = M_h + \Delta t^2 \left(\theta - \frac{1}{4}\right) K_h$.

3.2 Stability

The scheme (7) is shown to be stable if the modified mass matrix $\widetilde{M}_h$ is definite positive, which is the classical CFL condition of the θ-scheme. The treatment of nonlinear terms does not impact the stability properties.

3.3 Complexity

An interesting property of scheme (7) is that for each time step, knowing $(Q_h^{n-1}, Q_h^n, Z_h^{n-1/2})$, the computation of $(Q_h^{n+1}, Z_h^{n+1/2})$ only requires the evaluation of $\mathbb{H}(Q_h^n)$ and the solution of one linear system of size $N \times N_h^q + N_h^\zeta$.

4 Convergence analysis

Let $e_Q^n = Q_h^n - Q_h(t^n)$ and $e_Z^n = Z_h^n - Z_h(t^n)$ where Q_h, Z_h are solutions of (6). We will show that :

$$\|e_Q^n\|_p \leqslant T\mathcal{R} \quad \text{and} \quad \|e_Z^{n+1/2}\|_p \leqslant \mathcal{R}, \quad \text{where}$$

$$\mathcal{R} = e^{K^- T}\left[\frac{\Delta t}{\sqrt{M^-}} \sum_{j=1}^{n_0} \|\epsilon_{h,1}^j\|_p + \frac{\Delta t}{\sqrt{A^-}} \sum_{j=1}^{n_0} \|\epsilon_{h,2}^j\|_p\right]$$

and K^-, M^-, A^- are the strictly positive minimal eigenvalues of K_h, $\widetilde{M}_h$ and A_h which are supposed definite positive.
Truncation errors $\epsilon_{h,i}^j$ are $\mathcal{O}(\Delta t^2)$ if the semi-discrete solution is regular enough in time (Q_h must be $\mathcal{C}^4$ and Z_h must be $\mathcal{C}^3$).

5 Numerical illustration

A piano string is modeled (see [1]) and forced with a $\mathcal{C}^\infty$ in space and time compactly supported source. Fig. 1 shows the relative L^2 in space and L^∞ in time consecutive error between the solution computed with a time step Δt and the refined one with step $\Delta t/2$, using second order FEM. As expected the presented IEQ scheme shows quadratic convergence.

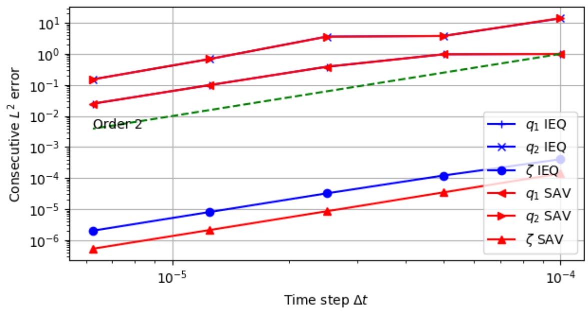

Figure 1: L^2 in space and L^∞ in time relative error of several schemes

6 Prospects

Another promising idea [3] is to define ζ as

$$\zeta(t) = \sqrt{2\int_0^L \mathcal{U}(\partial_x q)(x,t)dx + c} \qquad (8)$$

This auxiliary variable is now a scalar and the new system equivalent to (7) is only of size $N \times N_h^q + 1$. This SAV scheme is faster than IEQ and the relative error between the two obtained solutions is only 10^{-8}. Convergence is also quadratic (see Fig. 1).

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