Analysis of shape optimization problems for the Kuznetsov equation

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Abstract

In various biomedical applications, precise focusing of nonlinear ultrasonic waves is crucial for efficiency and safety of procedures. This work analyzes a class of shape optimization problems constrained by general quasi-linear acoustic wave equations that arise in high-intensity focused ultrasound (HIFU) applications and extends on the work of [3] on the Westervelt pressure equation. Within our theoretical framework, the Westervelt and Kuznetsov equations of nonlinear acoustics in potential form are obtained as particular cases. To prove the existence of the Eulerian shape derivative, we study the local well-posedness and regularity of the forward problem, uniformly with respect to shape variations. Additionally, we prove Hölder-continuity of the acoustic potential with respect to domain deformations in order to rigorously compute the shape derivative within the varitional framework of [1] for different cost-functionals of practical interest. The talk will be based on [2].

Keywords: nonlinear acoustics, shape optimization, Kuznetsov's equation, energy method, HIFU.

1 Introduction

High-Intensity Focused Ultrasound (HIFU) is emerging as one of the most promising non-invasive tools in treatments of various solid cancers. However, its wide-scale use hinges on the ability to guarantee the desired sound behavior in the focal region. Since these ultrasonic waves are commonly excited by one or several piezoelectric transducers arranged on a spherical surface, changes in their shape directly affect the propagation and focusing of sound waves and give rise to practically relevant optimization problems.

Motivated by this, in the present work we conduct the analysis of a class of shape optimization problems subject to a quasilinear wave model with general quadratic nonlinearities:

$$(1-2k\dot{\psi})\ddot{\psi} - c^2\Delta\psi - b\Delta\dot{\psi} - 2\sigma\nabla\dot{\psi}\cdot\nabla\psi = 0, \tag{1}$$

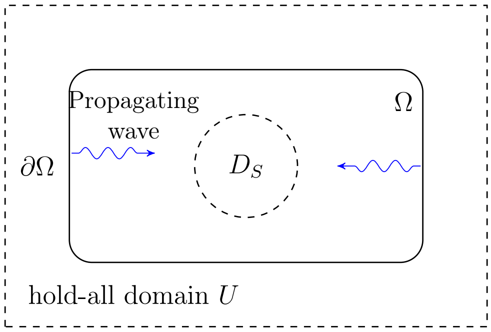

Figure 1: The optimization setup, where $D = D_S \times (t_0, t_1)$

on smooth spatial domains, assuming nonhomogeneous Neumann boundary excitation.

Besides, we not only treat the classical $L^2(L^2)$-tracking problem on $D = D_S \times (t_0, t_1)$, where the acoustic velocity potential ψ should match a desired output ψ_D on a given spatial focal region D_S within a certain time interval (t_0, t_1), i.e,

$$J(\psi,\Omega) = \frac{1}{2}\int_0^T \int_\Omega (\psi-\psi_D)^2\,\chi_{D_S}\,\mathrm{d}x\,\mathrm{d}s,$$

we also consider in this work an L^2-matching objective at final time:

$$J_T(\psi,\Omega) = \frac{1}{2}\int_\Omega (\psi(T)-\psi_{D_S})^2\,\chi_{D_S}\,\mathrm{d}x.$$

Additionally, we study an $L^2(L^2)$-tracking functional on $\dot{\psi}$, corresponding (up to a multiplicative constant) to tracking the sound pressure:

$$J_\mathrm{p}(\psi,\Omega) = \frac{1}{2}\int_0^T \int_\Omega \left(\dot{\psi}-f_D\right)^2 \chi_D\,\mathrm{d}x\mathrm{d}s;$$

see Figure 2 for an illustration of the optimization setup.

2 Well-posedness results

We study, under the hypothesis of small initial and boundary data, the well-posedness of the nonlinear Kuznetsov equation (1) coupled with inhomogeneous Neumann boundary excitation

$$\frac{\partial\psi}{\partial n} = g,$$

and initial data

$$\psi(0) = \psi_0 \in H^3(\Omega),\ \dot{\psi}(0) = \psi_1 \in H^2(\Omega).$$

g is taken in $H^2(H^{-1/2}(\partial\Omega)) \cap H^1(H^{3/2}(\partial\Omega))$ such that $\dot{g} \in L^\infty(H^{1/2}(\partial\Omega))$, and Ω is a $C^{2,1}$-regular domain.

Moreover, we study the well-posedness of the general adjoint problem

$$\begin{aligned} &\frac{\partial}{\partial t}\left((1-2k\dot{\psi})\dot{p}\right) - c^2\Delta p + b\Delta\dot{p} \\ &\qquad\qquad - 2\sigma\nabla\cdot(\dot{p}\nabla\psi) = f \\ &c^2\frac{\partial p}{\partial n} - b\frac{\partial \dot{p}}{\partial n} + 2\sigma g\dot{p} = 0 \\ &p(T) = p_0,\ \dot{p}(T) = p_1, \end{aligned} \tag{2}$$

which covers the different studied cost functionals for specific choices of f, p_0, and p_1; see [2] for details. We then establish sufficient conditions on f, p_0, and p_1 for which the adjoint problem is well posed and the shape derivative (given hereafter) is well defined.

3 Shape deformation and sensitivity

To describe shape variations and thus be able to express the shape derivative we use a vector field

$$h \in \mathcal{D} = \left\{ h \in C^{2,1}\left(\overline{U}, \mathbb{R}^l\right) \,\middle|\, h|_{\partial U} = 0 \right\},$$

to perturb the identity [4].

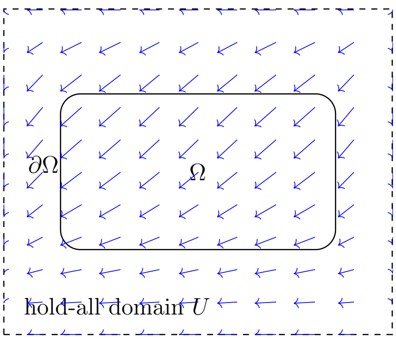

Figure 2: Pertubing the identity

Adopting the variational framework developed in [3], we rely on the Hölder continuity of the potential with respect to domain perturbations. We furthermore show that the Kuznetsov equation is uniformly well posed with regard to small enough shape variations.

Under the assumptions of this and previous sections, we then derive shape derivatives of the form required by the Delfour–Hadamard–Zolésio structure theorem.

Theorem 1 *The shape derivatives for cost functionals J, J_{p}, and J_T exist in the direction of any $h \in \mathcal{D}$ and are given by*

$$\begin{aligned} \mathrm{d}J(\Omega)h = &\int_0^T \int_{\partial\Omega} \Big(\frac{\partial}{\partial n}((c^2 g + b\dot{g})p) \\ &+ (c^2 g + b\dot{g})p\kappa\Big)(h\cdot n)\,\mathrm{d}\gamma\,\mathrm{d}s \\ &- \int_0^T \int_{\partial\Omega} \Big((1-2k\dot{\psi})\ddot{\psi}p + c^2\nabla p\cdot\nabla\psi \\ &+ b\nabla p\cdot\nabla\dot{\psi} - 2\sigma p\nabla\psi\cdot\nabla\dot{\psi}\Big)(h\cdot n)\,\mathrm{d}\gamma\,\mathrm{d}s, \end{aligned}$$

where κ stands for the mean curvature of $\partial\Omega$ and p depends on the choice of the cost-functional via (2).

4 Conclusions

In this work, we have analyzed shape optimization problems governed by general wave equations that model nonlinear ultrasound propagation and, as such, arise in HIFU applications. In particular, we have established sufficient conditions for the well-posedness and regularity of the underlying wave models with nonhomogeneous Neumann boundary conditions, uniformly with respect to shape deformations, as well as the Hölder continuity of the solutions. Furthermore, we have studied the corresponding adjoint problems and rigorously computed shape derivatives for several objectives of practical interest.

Future work will be concerned, among others, with the numerical analysis and simulation of the Kuznetsov equation and of the optimization problems presented in this work.

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