Recent Advances in Elastodynamics by Time-Domain Energetic Boundary Element Method

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Abstract

This contribution aims at illustrating some recent advances in the application of the Energetic Boundary Element Method (EBEM) to the numerical resolution of Elastodynamic problems both in exterior and interior domains.

Keywords: Elastodynamics, Energetic Boundary Element Method

1 Introduction

From the initial work of [1] and [2], it has been clear that the energetic weak approach to boundary integral formulations gives a good theoretical setting for the investigation of elastodynamic wave propagation.
The method is based on a boundary integral representation formula for the differential problem solution, then a weak formulation linked to the energy of the system is applied in order to achieve, in the approximation phase, accurate and stable numerical results.
However, the extension of the EBEM implementation is not straightforward since the computation of linear system entries requires sophisticated numerical strategies depending on the problem at hand (2D or 3D, bounded or an unbounded domain equipped with Dirichlet and/or Neumann conditions) and on the integral formulation used to represent the solution. In particular an accurate study of the numerical evaluation of double space integrals involved must be performed looking at the singularities of the integrand functions [3].
Further, the increasing complexity and dimensionality of test problems compels to perform a fast implementation by parallel computing on CPUs and GPUs and applying compression algorithms.

2 The model problem

Consider a domain $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$, then the displacement $u = (u_1, \dots, u_n)$ during the time interval $[0, T]$ in a linear, homogeneous, elastic and isotropic medium is described through the Navier equation by components:

$$\sum_{h,k,l=1}^{n} \frac{\partial}{\partial x_h}\left(\mathcal{C}_{ih}^{kl}\frac{\partial u_k}{\partial x_l}(x,t)\right) - \rho \ddot{u}_i(x,t) = 0$$
$$i = 1, \dots, n \qquad \forall (x,t) \in \Omega \times (0,T] \tag{1}$$

with mass density ρ and Hooke tensor $\mathcal{C}_{ih}^{kl}$ depending on the elastic material properties. The description of the problem is completed by initial and Dirichlet and/or Neumann boundary conditions

$$\begin{array}{ll} u(x,0) = 0; \quad \dot{u}(x,0) = 0 & x \in \Omega \\ u(x,t) = g_D(x,t) & x \in \Sigma_D := \Gamma_D \times [0,T] \\ p(x,t) = g_N(x,t) & x \in \Sigma_N := \Gamma_N \times [0,T] \end{array} \tag{2}$$

being $p_i(x,t) := \sum_{h,k,l=1}^{n} \mathcal{C}_{ih}^{kl}\frac{\partial u_k}{\partial x_l}(x,t) n_h(x)$ the i-th component of the traction p defined with respect to a normal vector n at a point x of the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$.

3 The energetic boundary integral formulation

Starting from the Somigliana identity and applying the arguments related to energy as done in scalar wave propagation problems [4], the unknown traction p on Γ_D and displacement u on Γ_N appear to be solutions of the system

$$\begin{array}{l} ((V_{ij}p_j), \dot{\phi}_i)_{L_2(\Sigma_D)} - ((K_{ij}u_j), \dot{\phi}_i)_{L_2(\Sigma_D)} = \\ \qquad (f_{D,i}, \dot{\phi}_i)_{L_2(\Sigma_D)} \\ ((D_{ij}u_j), \dot{\psi}_i)_{L_2(\Sigma_N)} - ((K'_{ij}p_j), \dot{\psi}_i)_{L_2(\Sigma_N)} = \\ \qquad (f_{N,i}, \dot{\phi}_i)_{L_2(\Sigma_N)} \end{array} \tag{3}$$

involving the fundamental solution tensor through the classical integral operators V, K, K', D as defined in [5]. ϕ defined on Σ_D and ψ defined on Σ_N are test functions belonging to the functional space of p and u respectively.

4 Numerical results

Preliminary numerical results have been published in [3] and in [6].
In the figure 1 there is the representation at

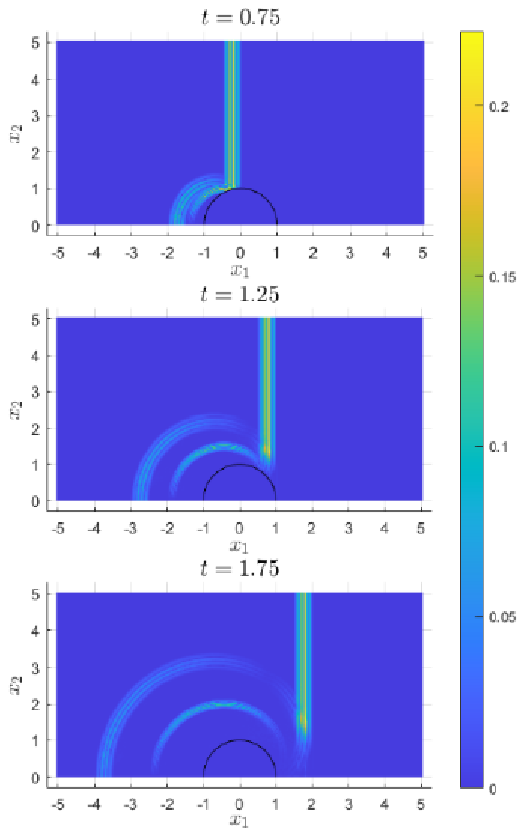

Figure 1: Example of 2D elastodynamic wave propagation.

some time instants of the intensity (Euclidean norm) of the total displacement generated by an incident plane pressure wave that propagates in a 2D domain in the horizontal direction from left to right. The incoming plane pressure wave u_{inc} propagates along the horizontal direction $\mathrm{k} = (1, 0)^T$ with phase velocity equal to 2 and impacts on the left side of the disk at the time instant $t = 0.25$ with the following shape

$$u_{inc}(x, t) = -\mathrm{k}g(c_P(t - 0.5) - \mathrm{k} \cdot \mathrm{x})$$

defined by g represented in Figure 2, setting the primary and the secondary waves velocities $c_P = 2$ and $c_S = 1$. The total displacement is obtained by summing to the incoming wave, the reflected wave that results solving the Dirichlet problem with datum $g_D(x, t) = -u_{inc}(x, t)$.

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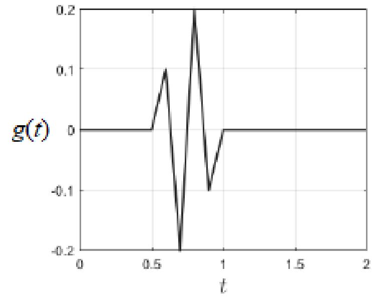

Figure 2: Shape of incident pressure wave.

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