Advances in overlapping-domain framework for wave propagation in heterogeneous and unbounded regions

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Abstract

Simulation of absorbed- and scattered-fields induced by an incident wave impinging on bounded heterogeneous configurations is fundamental for numerous applications. Scalar absorbed- and scattered-fields can be modeled by the Helmholtz partial differential equation (PDE) defined inside and exterior to the configuration, respectively. Finite element and boundary element methods (FEM and BEM), respectively, provide efficient tools for simulations of the Helmholtz PDE in the interior (varying wave speed) bounded domain and in its complement (constant wave speed) exterior medium. A novel coupled overlapping FEM-BEM framework was introduced for the interior-exterior Helmholtz model in [2] and for the 2D framework, FEM-BEM numerical analysis was developed in [3]. The main aim of the present work is on the advancement of the framework with numerical analysis for the 3D model.

Keywords: Helmholtz, heterogeneous, unboundedness, scattering, FEM-BEM

1 Heterogeneous and unbounded model

We consider the heterogeneous Helmholtz wave propagation problem for the total field u:

$$\Delta u + k^2 n^2\, u \;=\; 0, \quad \text{in } \mathbb{R}^3, \qquad (1)$$

with the Sommerfeld radiation condition (SRC) [1] for the scattered field $u^{\rm sc}$. In (1), $k > 0$ is a constant wavenumber and n is the spatially varying refractive index function with the variable wave speed restricted to a compact domain Ω_0. That is, n is a piecewise-continuous function with $n|_{\Omega_0^c} \equiv 1$. The unknown of the problem is the total field $u := u^{\rm sc} + u^{\rm inc}$, where typically $u^{\rm inc} = \exp({\rm i}k\widehat{\boldsymbol{d}}\cdot\boldsymbol{x})$ is an incident plane wave.

The majority of computational wave modeling approaches over the last five decades (see [6] and references therein) are restricted to either the FEM or BEM. This is because the literature is dominated by either truncating the above model to a bounded absorbing medium (and hence not satisfying the SRC exactly) or not allowing for practically important spatially varying wave speeds occurring inside a domain, say, $\Omega_0 \subset \mathbb{R}^d$ by assuming constant wave speeds [1]. Computational models that satisfy the SRC exactly facilitate accurate simulation of the far-field using high-order algorithms. Far fields are important for quantities of interests such as differential scattering cross sections (DSCS) [7]; and also for solving inverse wave propagation problems [1]. Our overlapping FEM-BEM algorithm satisfies the SRC exactly and also allow for heterogeneous refractive index in the Helmholtz model, consequently leading to simulations of the 3D heterogeneous media DSCS.

2 Overlapping-domain equivalent model

For practically incorporating both heterogeneity and unboundedness (H-U) in the full space $\mathbb{R}^d$ model, we developed and analyzed an equivalent continuous overlapping-domain equivalent model in both two and three dimensional settings. Our mathematically established equivalent continuous model approach in [2] is entirely different from the continuous (and discrete) models investigated using non-overlapping framework with a single interface coupling, for which several open challenging analysis problems remain to be solved (see [5, 8] and references therein). Advances in computational and numerical analysis counterpart of the full equivalent continuous model in [2] has been the aim of our recent and ongoing work. Below, we briefly introduce the continuous and a discrete framework. The focus of our present work is on algorithm, numerical analysis, and implementation using high-order discretization of the H-U overlapping construction for the 3D case. The continuous H-U framework in [2] was based on introducing two, free to choose, artificial boundaries: a smooth closed surface surrounding Ω_0, with an exterior free-space, and a further outer simple polyhedral boundary Indeed, let Q be a polyhedron with $\Omega_0 \subset Q$ and let $S \subset Q \setminus \overline{\Omega}_0$ be the smooth closed surface. Thus, Q includes the full model heterogeneity and intersects with a bounded region exterior to S, and the unbounded exterior to the curved boundary S is a constant wave speed medium. Hence, in our computational framework, we can apply high-order FEM for the total wave u in the interior heterogeneous problem in Q, and use spectrally accurate BEM for the exterior model [1,4], that are specially designed for curved boundaries for the scattered field u^{sc} and that exactly satisfies the SRC in the BEM model as well.

3 The FEM-BEM coupling procedure

We conclude by briefly describing our algorithm. For the interior problem, we seek approximations in V_h, a continuous high-order FEM space on triangulated conformal meshes of Q. Let

$$V_h^H = \{u_h \in V_h \ : \ \Delta_h u_h + k^2 n^2 u_h =_h 0\}$$

(i.e, elements of V_h which are FE solutions of the Helmholtz equation). Then we have

$$V_h^H \ni u_h \approx u|_Q$$

simply by demanding the trace relation:

$$\gamma_\Sigma u_h = \gamma_{\Sigma,h} u = \gamma_{\Sigma,h} u^{\text{sc}} + \gamma_{\Sigma,h} u^{\text{inc}}, \qquad (2)$$

with $\Sigma = \partial Q$, the boundary of Q and $\gamma_{\Sigma,h}$ above being the result of interpolation of $\gamma_\Sigma u$, the trace of u on Σ.

On the other hand, we seek spectrally accurate scattered field BEM solutions in $\mathbb{T}_N$, a discrete space of smooth functions on S with a maximum degree parameter N [1,4] such that

$$\mathbb{T}_N \ni \varphi_N \text{ s.t. } \mathcal{L}_{k,N}\varphi_N \approx u^{\text{sc}}.$$

In the expression above, $\mathcal{L}_{k,N}$ is a (discrete evaluation of a) robust layer potential representation of radiation solutions of Hemlholtz equation and φ_N a suitable density determined by

$$\gamma_{S,N}(\mathcal{L}_{k,N}\varphi_N - u^{\text{sc}}) = 0. \qquad (3)$$

That is, φ_N is the spectrally accurate surface density solution of a BIE.

Gathering (2)-(3), we conclude that (φ_N, $\gamma_\Sigma u_h$), the ultimate true unknowns of our computational framework, is solution of

$$\begin{aligned} \gamma_\Sigma u_h - \gamma_{\Sigma,h}(\mathcal{L}_{k,N}\varphi_N) &= \gamma_{\Sigma,h} u^{\text{inc}}, \\ -\gamma_{S,N} u_h + \gamma_{S,N}(\mathcal{L}_{k,N}\varphi_N) &= -\gamma_{S,N} u^{\text{inc}}. \end{aligned}$$

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