Eddy-current asymptotics of the Maxwell PMCHWT formulation: the multi-body case

Edouard Demaldent[1,*], Marc Bonnet[2]

[1]Université Paris-Saclay, CEA, List, F-91120, Palaiseau, France
[2]POEMS, ENSTA Paris, 91120 Palaiseau, France

*Email: edouard.demaldent@cea.fr

Abstract. A low-frequency, high-conductivity asymptotic approximation of the Maxwell transmision problem for configurations with highly-conducting, moderately-conducting and non-conducting bodies, motivated by eddy current testing applications, is proposed and validated.

Keywords: Maxwell equations, PMCHWT formulation, eddy currents, asymptotic expansion

Introduction In eddy current (EC) testing applications, ECs $\sigma\mathbf{E}$ ($\mathbf{E}$: electric field, σ: conductivity) are induced in tested metal parts by a low-frequency (LF) source idealized as a closed current loop in air. The EC model corresponds to the magneto-quasi-static approximation of the Maxwell problem, which neglects the displacement current. In presence of highly conductive (HC) media, the boundary integral equation (BIE) of the first kind under the magneto-quasi-static approximation proposed in [4] was shown in [2] to coincide with the leading order of an asymptotic expansion of the Maxwell BIE in a small parameter γ reflecting both LF and HC assumptions. Here, we extend [2] and derive a low-γ asymptotic approximation for configurations involving multiple moderately-conducting (MC, $\sigma = O(1)$) or non-conducting (NC) objects in addition to HC objects, which may be multiply-connected and nested.

Setting We consider the time-harmonic electromagnetic transmission problem whereby M objects with conductivity σ_a, complex permittivities $\varepsilon_a := \varepsilon_0\varepsilon_{\mathrm{r}a} = \varepsilon^{\mathrm{d}}_a + \mathrm{i}\sigma_a/\omega$ and permeabilities $\mu_a = \mu_0\mu_{\mathrm{r}a}$ $(1 \le a \le M)$, which occupy the bounded Lipschitz domains $\Omega_a \subset \mathbb{R}^3$, are surrounded by vacuum filling $\Omega_0 := \mathbb{R}^3 \setminus (\overline{\Omega_1} \cup \ldots \cup \overline{\Omega_M})$ (Fig. 2). The objects are excited by fields created by a given current density $\boldsymbol{J}_{\mathrm{s}}$ with compact support in Ω_0. The electric field $\boldsymbol{E}$ thus solves the transmission problem

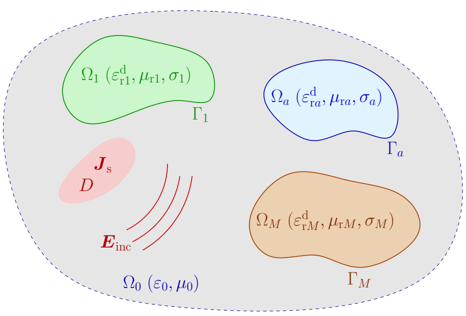

Figure 1: Scattering by multiple objects.

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\begin{aligned}
(\mathbf{rot\,rot} - \kappa_0^2)\boldsymbol{E} &= \mathrm{i}\omega\mu_0\boldsymbol{J}_{\mathrm{s}} && \text{in } \Omega_0,\\
(\mathbf{rot\,rot} - \kappa_a^2)\boldsymbol{E} &= \boldsymbol{0} && \text{in } \Omega_a,\\
\boldsymbol{\gamma}^{a-}_{\times}\boldsymbol{E}_a - \boldsymbol{\gamma}^{a+}_{\times}\boldsymbol{E} &= \boldsymbol{0} && \text{on } \Gamma_a, \qquad (1)\\
\mu_{\mathrm{r}a}^{-1}\boldsymbol{\gamma}^{a-}_{N}\boldsymbol{E} - \boldsymbol{\gamma}^{a+}_{N}\boldsymbol{E}_0 &= \boldsymbol{0} && \text{on } \Gamma_a,\\
\boldsymbol{E} \text{ radiating at infinity,} & && (1 \le a \le M)
\end{aligned}
$$

wherein $\kappa_0^2 = \varepsilon_0\mu_0\omega^2$, $\kappa_a^2 = \kappa_0^2\big(\varepsilon^{\mathrm{d}}_{\mathrm{r}a} + \mathrm{i}\sigma_a/\omega\varepsilon_0\big)\mu_{\mathrm{r}a}$ are the wavenumbers in Ω_a, and the trace operators $\boldsymbol{\gamma}^{a\pm}_{\times}$, $\boldsymbol{\gamma}^{a\pm}_{N}$ are defined by $\boldsymbol{\gamma}^{a\pm}_{\times}\boldsymbol{u} := \boldsymbol{\gamma}^{a\pm}\boldsymbol{u}\times\boldsymbol{n}$ and $\boldsymbol{\gamma}^{a\pm}_{N}\boldsymbol{u} := \boldsymbol{\gamma}^{a\pm}_{\times}\mathbf{rot}\,\boldsymbol{u}$ in terms of the exterior and interior Dirichlet traces $\boldsymbol{\gamma}^{a\pm}$ on Γ_a. Problem (1) is then recast as the PMCHWT [5] integral equation system, whose primary unknowns are the tangential current densities $\boldsymbol{J} \in \boldsymbol{\mathcal{V}}^a$ (electric) and $\boldsymbol{M} \in \boldsymbol{\mathcal{V}}^a$ (magnetic) on each Γ_a (with $\boldsymbol{\mathcal{V}}^a := \big\{\boldsymbol{v} \in (\boldsymbol{\mathcal{U}}^a)' : \mathrm{div}_S\boldsymbol{v} \in H^{-1/2}(\Gamma_a)\big\}$ in terms of $\boldsymbol{\mathcal{U}}^a := \boldsymbol{n}\times\big(\boldsymbol{\gamma}^{a-}\boldsymbol{H}^1(\Omega_a)\times\boldsymbol{n}\big)$). Moreover, a Helmholtz–Hodge decomposition [3] $\boldsymbol{\mathcal{V}}^a = \boldsymbol{\mathcal{V}}^a_{\mathrm{L}} \oplus \boldsymbol{\mathcal{V}}^a_{\mathrm{T}}$ of each $\boldsymbol{\mathcal{V}}^a$ is used, where $\boldsymbol{\mathcal{V}}^a_{\mathrm{L}} := \big\{\boldsymbol{u} \in \boldsymbol{\mathcal{V}}^a : \mathrm{div}_S\boldsymbol{u} = 0\big\}$, inducing additive decompositions $\boldsymbol{J}^a = \boldsymbol{J}^a_{\mathrm{L}} + \boldsymbol{J}^a_{\mathrm{T}}$ and $\boldsymbol{M}^a = \boldsymbol{M}^a_{\mathrm{L}} + \boldsymbol{M}^a_{\mathrm{T}}$. The weak formulation of the PMCHWT integral problem has the form: find $\mathbb{X} \in \mathbb{V}$,

$$
\big\langle \widetilde{\mathbb{X}}, \boldsymbol{\gamma}_{\times}\mathbb{Z}\mathbb{X}\big\rangle = \big\langle \widetilde{\mathbb{X}}, \boldsymbol{\gamma}_{\times}\mathbb{Y}\big\rangle \quad \text{for all } \widetilde{\mathbb{X}} \in \mathbb{V} \qquad (2)
$$

where $\mathbb{Z}$ is a $4M \times 4M$ operator matrix, $\mathbb{X}$ and $\mathbb{V}$ concatenate the unknowns $\boldsymbol{J}^a_{\mathrm{L}}, \boldsymbol{J}^a_{\mathrm{T}}, \boldsymbol{M}^a_{\mathrm{L}}, \boldsymbol{M}^a_{\mathrm{T}}$ on each Γ_a and the associated spaces, and $\widetilde{\mathbb{X}}$ are correspondingly defined test functions. The boundary element discretization of (2) uses the *loop–tree* decomposition, see e.g. [1].

Expansion of the PMCHWT system We introduce the dimensionless parameter $\gamma := \kappa_0 L$, with L a characteristic length. Each object Ω_a is taken as either NC (i.e. $\sigma_a = 0$), MC (i.e. $\sigma_a = c_a\sigma_{\mathrm{ref}}$) or HC (i.e. $\sigma_a = \gamma^{-1}c_a\sigma_{\mathrm{ref}}$), each dimensionless factor c_a being fixed (i.e. independent on γ) and with $\sigma_{\mathrm{ref}} := L^{-1}\sqrt{\varepsilon_0/\mu_0}$. In particular, the HC bodies are in the eddy current regime. Our general aim is to define approximate solutions of (2) for low values of γ by seeking an expansion in powers of γ of $\mathbb{X}$.

For the surface current densities on each Γ_a, we obtain formal expansions of the form

$$\begin{aligned} \boldsymbol{J}^a_{\mathrm{L}} &= \widehat{\boldsymbol{J}}^a_{\mathrm{L},0} + \gamma \widehat{\boldsymbol{J}}^a_{\mathrm{L},1} + \gamma^{3/2} \widehat{\boldsymbol{J}}^a_{\mathrm{L},3/2} + O(\gamma^2), \\ \gamma^{-2} \boldsymbol{J}^a_{\mathrm{T}} &= \widehat{\boldsymbol{J}}^a_{\mathrm{T},0} + \gamma \widehat{\boldsymbol{J}}^a_{\mathrm{T},1} + \gamma^{3/2} \widehat{\boldsymbol{J}}^a_{\mathrm{T},3/2} + O(\gamma^2), \\ \boldsymbol{M}^a_{\mathrm{L}} &= \widehat{\boldsymbol{M}}^a_{\mathrm{L},0} + \gamma \widehat{\boldsymbol{M}}^a_{\mathrm{L},1} + \gamma^{3/2} \widehat{\boldsymbol{M}}^a_{\mathrm{L},3/2} + O(\gamma^2), \\ \boldsymbol{M}^a_{\mathrm{T}} &= \widehat{\boldsymbol{M}}^a_{\mathrm{T},0} + \gamma \widehat{\boldsymbol{M}}^a_{\mathrm{T},1} + \gamma^{3/2} \widehat{\boldsymbol{M}}^a_{\mathrm{T},3/2} + O(\gamma^2), \end{aligned} \tag{3}$$

with all coefficients $\widehat{\boldsymbol{J}}^a_{\mathrm{L},0}$ defined as solutions of integral problems arising from the expansion of problem (2). This in turn results in expansions

$$\begin{aligned} \boldsymbol{E} &= \gamma\big(\boldsymbol{E}_0 + \gamma \boldsymbol{E}_1 + \gamma^{3/2} \boldsymbol{E}_{3/2} + O(\gamma^2)\big) \\ \boldsymbol{H} &= \boldsymbol{H}_0 + \gamma \boldsymbol{H}_1 + \gamma^{3/2} \boldsymbol{H}_{3/2} + O(\gamma^2) \\ \Delta Z &= \gamma\big(\Delta Z_0 + \gamma \Delta Z_1 + \gamma^{3/2} \Delta Z_{3/2} + O(\gamma^2)\big), \end{aligned} \tag{4}$$

with all coefficients well-defined in terms of those of (3), for the electromagnetic fields $\boldsymbol{E}, \boldsymbol{H}$ in Ω_0 and in each Ω_a and the impedance variation ΔZ (see [2, eq. 27], of main interest in EC testing).

Remarks and extensions The leading-order operator matrix $\widehat{\mathbb{Z}}_0$ arising from the expansion of problem (2) has some zero blocks entries, such that any system of the form $\widehat{\mathbb{Z}}_0 \widehat{\mathbb{X}} = \widehat{\mathbb{Y}}$ can be solved blockwise in three stages, with computational benefits. Stage 1 yields $(\widehat{\boldsymbol{J}}^a_{\mathrm{L}}, \widehat{\boldsymbol{M}}^a_{\mathrm{L}}, \widehat{\boldsymbol{M}}^a_{\mathrm{T}})_{a \in \mathrm{HC}}$ and $(\widehat{\boldsymbol{J}}^a_{\mathrm{L}}, \widehat{\boldsymbol{M}}^a_{\mathrm{T}})_{a \in \mathrm{NC} \cup \mathrm{MC}}$ at any order and often suffices in practice as it provides at order γ^0 the correct leading-order approximations of ΔZ, of $\boldsymbol{H}$ everywhere, and of $\boldsymbol{E}$ in the HC bodies.

Moreover, the operator submatrix for Stage 1 is found to not depend on σ_a in the MC bodies, if any. Treating the latter as NC thus produces the same Stage-1 solutions, and hence all the correct leading-order approximations obtainable from Stage 1 at order γ^0 alone.

In the absence of MC bodies, all terms of orders 1 and 3/2 vanish in (3) and (4), so that (for example) we have $\Delta Z = \gamma\big(\Delta Z_0 + O(\gamma^2)\big)$.

The asymptotic formulation outlined above for homogeneous and simply-connected objects has in addition been extended for also catering to (i) bi-material objects of different conductivity classes (e.g. a HC object embedded in a MC object) and (ii) multiply connected objects (setting the global loop functions apart due to their distinct behavior under integral operators). Both cases required significant modifications of the asymptotic expansion.

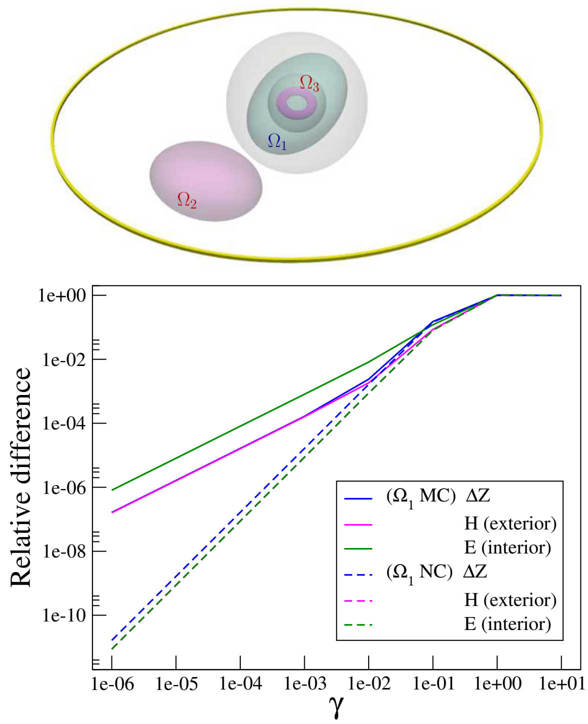

Figure 2: Multiple-body case with bi-material object: configuration (top), relative differences with full Maxwell (bottom). Ω_3 (HC) embedded in Ω_1 (either NC or MC), Ω_2 is HC. The grey spheres inside and outside Ω_1 are evaluation surfaces.

Numerical example We validate the asymptotic approximations (3), (4) on an example involving two objects (one bi-material, one homogeneous), the embedded object being multiply connected (Figure 2). The predicted convergence orders are confirmed. Other examples of this type, as well as applications to EC testing configurations, will be presented.

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