A trace theorem on an infinite b-adic tree

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Abstract

We discuss the construction of the trace operator defined on Sobolev spaces over a class of infinite radial trees based on an identification of the abstract boundary with an Euclidean domain. The case of dyadic trees was covered by Maury, Salort, Vannier in [2], and we adapt their constructions and results in order to include to the case of b-adic trees with arbitrary $b \geq 2$. While the overall approach is very similar, one has to introduce several new steps related to the discrete Fourier transform and to a more careful analysis of various characterizations of Besov and Sobolev spaces.

Keywords: Trace theorems, Sobolev spaces, Fractal graph, Boundary values

1 Motivation

A wave propagation or diffusion process inside an object occupying some region in the space is typically described by a suitable differential or finite-difference equation, while the interaction with the surrounding is taken into account using a boundary or transmission condition. In many cases, one looks for solutions of boundary value problems in suitable Sobolev spaces, and the regularity of boundary data (=traces) implies important conclusions on the existence and the regularity of the solutions. While there is an established trace theory for Sobolev spaces over smooth domains, many important questions remain open for more involved geometric objects (in particular, for the objects that become "degenerate" near the boundary, even the rigorous definition and the existence of boundary traces represent a difficult problem). In the present work, we address the related questions for a class of fractal objects represented by Laplace-type operators on special self-similar trees, which is a natural extension of the mathematical model of respiratory systems introduced in [2] and can be viewed as a discrete approximation to the study of wave propagation in thin networks proposed in [1].

2 Resistive trees

The infinite b-adic tree T with root o is shown in Figure 1. Its internal properties will be taken into account using an additional parameter $\alpha \in (0, b)$ as follows: the resistance $r(x, y)$ of an edge xy in the nth generation is α^n. Denote by $V(T)$ the set of all vertices of T, then the first Sobolev space is defined as

$$H^1(T) = \Big\{ p : V(T) \to \mathbb{C} : \\ |p|^2 := \sum_{x\sim y} \frac{|p(x) - p(y)|^2}{r(x, y)} < \infty \Big\}$$

equiped with the norm $\|p\|^2 := |p|_1^2 + |p(0)|^2$. The H^1-closure of the set of p with finite supports will be denoted as $H_0^1(T)$. It is straightforward to show the decomposition

$$H^1(T) = H_0^1(T) \oplus H_\Delta^1(T),$$

where $H_\Delta^1(T)$ is the null space of a linear operator (a kind of discrete laplacian) Δ associated with T.

It is natural to interpret $H_0^1(T)$ as the set of functions vanishing at the boundary and the quotient H^1/H_0^1 as the space of functions at the boundary, which then reduces to the study of boundary values of the "harmonic functions", i.e. of the elements of H_Δ^1.

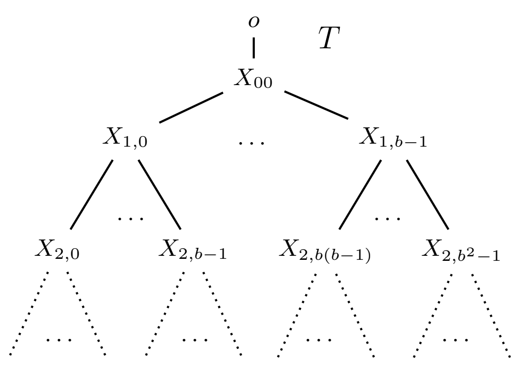

Figure 1: Structure of a b-adic tree

3 Abstract trace operator

The abstract trace operator is constructed using a suitable basis in H_Δ^1. For that, the tree is decomposed in a special way.

Each point of the boundary can be naturally identified with an infinite sequence (c_n) with $c_n \in (0, 1, \dots, b-1)$, which corresponds to an infinite path from the root through the vertices X_{n,k_n}, with $k_n = \sum_{i=1}^{n} c_i b^{n-i}$. The set of such paths passing through $X_{n,k}$ will be denoted by $C_{n,k}$. If one identifies the whole boundary with a set Ω (having a finite volume $|\Omega|$), then one can naturally expects that the portion $\Omega_{n,k}$ identified with $C_{n,k}$ has the volume $b^{-n}|\Omega|$.

Denote by $T_{n,k}$ the infinite subtree composed of $X_{n,k}$ and all its children. Using an adapted discrete Fourier transform one is able to show that there exists a basis $(\varphi_0, \varphi_{n,k}^{(s)})$ in H^1_Δ such that $\varphi_0 = \mathbb{1}$ and $\varphi_{n,k}^{(s)}$ vanish outside $T_{n,k}$ and are radial along $T_{n,k}$. The main idea for the construction of the trace operator is that the trace of $\varphi_{n,k}$ is identified with the (suitably normalized) indicator function of $\Omega_{n,k}$.

4 Embedding

In order to implement rigorously the above idea, one needs additional assumptions on the subsets $\Omega_{n,k}$. Namely, assume that $\Omega \subset \mathbb{R}^d$ is a bounded connected Lipschitz domain and $\Omega_{n,k} \subset \Omega$ with $n \in \mathbb{N}$ and $k = 0, \dots, b^n - 1$ be its measurable subsets building a kind of multiscale decomposition in the following sense:

$$\bigcup_k \overline{\Omega_{n,k}} = \overline{\Omega} \text{ for all } n,$$

$$\bigcup_j \overline{\Omega_{n+1,bk+j}} = \overline{\Omega_{n,k}} \text{ for all } n, k,$$

$$|\Omega_{n,k}| = b^{-n}|\Omega|,$$

$$(\Omega_{n,k})_{k=0}^{b^n-1} \text{ are disjoint for all } n;$$

and satisfying some additional (explicitly formulated) assumptions ensuring that $\Omega_{n,k}$ do not "degenerate" (i.e. $\Omega_{n,k}$ should become "small" in all directions as n becomes large). We give explicit examples of suitable decompositions in every dimension.

If one accepts the above approach and hypotheses (which then reduces to the identification of $C_{n,k}$ with $\Omega_{n,k}$), then the "reasonable" trace operator γ is uniquely defined by the identifications

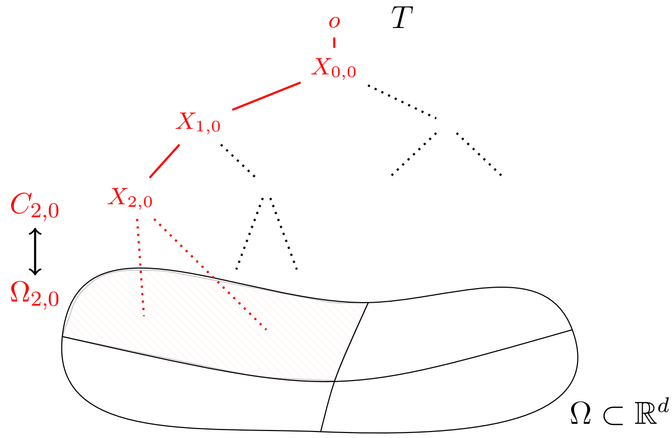

Figure 2: Embedding for $b = 2$: $C_{2,0} \leftrightarrow \Omega_{2,0}$

$$\gamma^\Omega(\varphi_{n,k}^{(s)})^2(x) = \begin{cases} \frac{e^{\frac{4\pi i s j}{b}} \alpha^{n+1}}{b-\alpha} & \text{if } x \in \Omega_{n+1,bk+j}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma^\Omega(\varphi_0)^2(x) = \mathbb{1}_\Omega, \quad \gamma^\Omega(H^1_0(T)) = 0.$$

Specifically, the Sobolev regularity of these trace functions can then be studied and we obtain our main result:

Theorem 1 *Denote*

$$s = \frac{d}{2}\left(1 - \frac{\ln \alpha}{\ln b}\right),$$

then

$$\gamma^\Omega(H^1(T)) = H^s(\Omega) \text{ if } s < \frac{1}{2},$$

$$\gamma^\Omega(H^1(T)) \hookrightarrow H^{s'}(\Omega) \text{ if } s \geq \frac{1}{2} \text{ and } s' < \frac{1}{2}.$$

If time permits, we will discuss an ongoing work dealing with an extension of the above approach to the trace theory of some self-similar metric graphs recently discussed in [1, 3].

References

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