Propagation of acoustic and gravity waves in the ocean: a new derivation for a general model

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Abstract

In this work we are interested in the modeling of propagation of acoustic and gravity wave in the ocean. An original model is obtained by the linearization of the compressible Navier-Stokes equations with free surface written in Lagrangian coordinates. We present an asymptotic analysis for small Mach number allowing to compare the obtained model to existing linear models.

Keywords: Fluid mechanics, Free-surface flow, Modeling, Acoustic-gravity waves

1 Introduction

Several authors have proposed to use the propagation of acoustic-gravity waves in the ocean to detect tsunamis [2], as the sound travels much faster in water than the tsunami wave itself.

To model the acoustic-gravity waves, we consider the Navier-Stokes equations for an iviscid, weakly compressible and free-surface fluid and aim at a linear approximation of these nonlinear equations that retains the two types of waves.

The ocean is supposed to be close to an equilibrium, stratified, with a varying density ρ and temperature T. The Navier-Stokes equations describing this state are written in Lagrangian coordinates for a rigorous treatment of the free surface. The system is then linearized around the state at rest, and a wave-like equation is obtained (it shares similarities with the Galbrun equation [4]). We compare our system with existing models in two ways: first we perform an asymptotic analysis for small Mach numbers and recover the equations governing an incompressible flow with free surface. Second we put the model back to eulerian coordinates and compare it with other well-known and widely used linear models [1, 3].

2 Lagrangien description

We start with the Navier-Stokes equations in Lagrangian coordinates. Let Ω be the domain of the ocean at rest, with boundary Γ_b at the bottom and horizontal boundary Γ_s at the surface. Let $\mathbf{d}$ be the displacement from Ω to the deformed configuration. The gradient of $\mathbf{d}$ is F, its Jacobian J and the fluid velocity $\mathbf{U} = \partial_t \mathbf{d}$. The Navier-Stokes equations are then

$$\partial_t \rho + \frac{\rho}{|J|} \nabla_\xi \cdot (|J| F^{-1} \mathbf{U}) = 0 \quad \text{in } \Omega, \tag{1}$$

$$\rho \partial_t \mathbf{U} + F^{-T} \nabla_\xi p = \rho \mathbf{g} \quad \text{in } \Omega, \tag{2}$$

with the boundary conditions, p^a being the atmospheric pressure and $\mathbf{f}$ a source term,

$$p = p^a \text{ on } \Gamma_s \quad \text{and} \quad \mathbf{U} \cdot \mathbf{n}_b = \mathbf{f} \text{ on } \Gamma_b. \tag{3}$$

The pressure and the density must satisfy the state law $p = f_p(\rho, T)$ with the temperature assumed given. With this law and energy conservation, an equation for the pressure is obtained

$$\partial_t p + \frac{\rho c^2}{|J|} \nabla_\xi \cdot (|J| F^{-1} \mathbf{U}) = 0. \tag{4}$$

Where c is the speed of sound in the fluid.

3 Linearization and the limit state

The equations (1)-(4) are linearized around a steady state for the ocean at rest: there is no mean current and the pressure, density and temperature have only vertical variations. This corresponds to an asymptotic expansion with $\epsilon \ll 1$:

$$\mathbf{d} = \epsilon \mathbf{d}_1 + \mathcal{O}(\epsilon^2), \quad \rho = \rho_0 + \epsilon \rho_1 + \mathcal{O}(\epsilon^2), \\ p = p_0 + \epsilon p_1 + \mathcal{O}(\epsilon^2). \tag{5}$$

Using these expressions in Eq.(1)-(4) and separating powers of ϵ yields a limit system and a system for the first order corrections.

For limit terms, one finds that the pressure p_0 and density ρ_0 depend only on the vertical coordinate ξ^3, with

$$p_0'(\xi^3) = -\rho_0 g, \qquad p_0 = p^a \text{ on } \Gamma_s \tag{6}$$

They must also satisfy $p_0 = f_p(\rho_0, T_0)$ where T_0 is the initial temperature. These equations are compatibility conditions between the choice of a state at rest and the hypothesis that this is a stable equilibrium.

4 A wave equation for the velocity

The system for the correction terms is written in Ω and reads

$$\rho_0 \frac{\partial \mathbf{U}_1}{\partial t} + \nabla_\xi p_1 - (\nabla_\xi \mathbf{d})^T \, \nabla_\xi p_0 = \rho_1 \mathbf{g}, \quad (7)$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla_\xi \cdot \mathbf{U}_1 = 0, \quad (8)$$

$$\frac{\partial p_1}{\partial t} + \rho_0 c_0^2 \, \nabla_\xi \cdot \mathbf{U}_1 = 0, \quad (9)$$

with boundary conditions

$$p_1 = 0 \text{ on } \Gamma_s \quad \text{and} \quad \mathbf{U}_1 \cdot \mathbf{n}_b = \mathbf{f} \text{ on } \Gamma_b. \quad (10)$$

Differentiating in time (7) and replacing ρ and p with (8), (9) we obtain a second order equation for $\mathbf{U}_1$,

$$\rho_0 \frac{\partial^2 \mathbf{U}_1}{\partial t^2} - \nabla_\xi \left(\rho_0 c_0^2 \nabla_\xi \cdot \mathbf{U}_1 \right) - (\nabla_\xi \mathbf{U}_1)^T \, \rho_0 \mathbf{g} + \rho_0 \nabla_\xi \cdot \mathbf{U}_1 \; \mathbf{g} = 0. \quad (11)$$

We introduce for the weak formulation the space

$$\mathcal{V} = \{ \mathbf{U}_1 \in H(\mathrm{div}, \Omega) \mid \mathbf{U}_1 \cdot \mathbf{n} \in L^2(\partial\Omega) \text{ and } \mathbf{U}_1 \cdot \mathbf{n}_b = 0 \text{ on } \Gamma_b \}. \quad (12)$$

The variational formulation of (11) is not straightforward to obtain but reads:

Find $\mathbf{U}_1 \in C^1([0,T]; L^2(\Omega)^3) \cap C^0([0,T]; \mathcal{V})$ solution to

$$\frac{d^2}{dt^2} (\mathbf{U}_1, \mathbf{V})_{\mathcal{H}} + a(\mathbf{U}_1, \mathbf{V}) = L(\mathbf{V}) \quad \forall \, \mathbf{V} \in \mathcal{V}, \quad (13)$$

where a is a symmetric bilinear form

$$\begin{aligned} a(\mathbf{U}_1, \mathbf{V}) = & \int_{\Gamma_s} \rho_0 \, g \mathbf{U}_1 \cdot \mathbf{e}_3 \mathbf{V} \cdot \mathbf{e}_3 \, \mathrm{d}s \\ & + \int_\Omega \rho_0 \left(c_0 \nabla_\xi \cdot \boldsymbol{\psi} - \frac{g}{c_0} (\mathbf{U}_1 \cdot \mathbf{e}_3) \right) \\ & \quad \times \left(c_0 \nabla_\xi \cdot \mathbf{V} - \frac{g}{c_0} (\mathbf{V} \cdot \mathbf{e}_3) \right) \mathrm{d}\xi \\ & + \int_\Omega N^2 \, \rho_0 \, \mathbf{U}_1 \cdot \mathbf{e}_3 \, \mathbf{V} \cdot \mathbf{e}_3 \, \mathrm{d}\xi, \quad (14) \end{aligned}$$

with the Brunt-Väisälä frequency N^2 defined by

$$N^2(\xi_3) = - \left(\frac{g^2}{c_0(\xi^3)^2} + g \frac{\rho_0'(\xi^3)}{\rho_0(\xi^3)} \right),$$

and L is a linear form

$$L(\mathbf{V}) = \int_\Omega \rho_0 \mathbf{V} \cdot \mathbf{f} \, \mathrm{d}\xi.$$

For values of ρ_0, ρ_0' and c_0 in the literature, N^2 is positive and so is the bilinear form a.

5 Asymptotic analysis for small Mach number

All the quantities are put in non-dimensional form. Introducing the non-dimensional number $\delta = Ma/Fr$, we obtain the equation

$$\rho_0 \frac{\partial^2 \mathbf{U}_1}{\partial t^2} - \frac{1}{\delta^2} \nabla_\xi \left(\rho_0 c_0 \nabla_\xi \cdot \mathbf{U}_1 \right) - (\nabla_\xi \mathbf{U}_1)^T \, \rho_0 \mathbf{g} + \rho_0 \nabla_\xi \cdot \mathbf{U}_1 \; \mathbf{g} = 0. \quad (15)$$

An asymptotic expansion for $\delta \ll 1$ is carried out. The limit solution is divergence free. In the limit equation the divergence term is replaced by a Lagrange multiplier. When $\mathbf{f} = 0$ and the density is constant, the Lagrange multiplier φ is solution to

$$\Delta \varphi = 0 \text{ on } \Omega \;, \; \partial_{tt} \varphi - \partial_3 \varphi = 0 \text{ on } \Gamma_s. \quad (16)$$

In other cases (density not constant) the obtained limit equation can be seen as a generalization of the classical Cauchy-Poisson equation for incompressible flow.

6 Comparison with linear models in Eulerian coordinates

The first order system is written in Eulerian coordinates. The set of equations of [1] is obtained. Compared to [3], our model retains more terms because of two modelling choices: the fluid is not assumed irrotational, and the density depends not only on the pressure but also on the temperature. The interest of our model is demonstrated by numerical simulations.

References

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