High-frequency estimates and error bounds on the h-BEM for the Helmholtz exterior Neumann problem

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Abstract

In this talk, we consider solving the Helmholtz equation posed in the exterior of a smooth obstacle, with Neumann boundary conditions, and using second-kind boundary integral equations (BIEs). We study the h-version of the Galerkin boundary element method, and its accuracy at high-frequency.

The study of the behaviour of Galerkin error when the frequency increases requires, in particular, high-frequency estimates for the considered boundary integral operators. In the case of Dirichlet boundary condition, results are well-known in the literature for the standard second-kind BIE, but in the case of Neumann boundary condition, where regularisation techniques are commonly used, such estimates are rare.

Our contribution is twofold: we first present high-frequency estimates for a particular regularised formulation in the case of Neumann boundary conditions, and then, we present a frequency-explicit bound for its discretisation error.

Keywords: boundary integral equations, high-frequency, Helmholtz, Neumann boundary condition

1 Introduction

We are interested in scattering problems with a smooth obstacle, and we consider solving the Helmholtz equation $\Delta u + k^2 u = 0$ using second-kind BIEs posed in $L^2(\Gamma)$, where Γ is the boundary of the obstacle. We consider the h-version of the Galerkin boundary element method, i.e., we consider a sequence of piecewise-polynomial approximation spaces with fixed polynomial degree p and decreasing h.

The focus of our work is to understand how quickly h needs to decrease with k to maintain accuracy of the Galerkin solution as $k \to \infty$. For the finite element method, numerous works have been published since the seminal work of [1], and it is still the subject of ongoing research. In the context of the boundary element method, there has been fewer investigations of this question, and they all focused on the Dirichlet problem.

A standard approach for solving scattering problems is to use “combined-field” BIEs, which have the advantage to be well-posed for every $k > 0$. Then, an intrinsic difficulty of the Neumann problem is the presence of the hypersingular operator, which is an operator of order 1, so that the associated boundary integral operator is not bounded from $L^2(\Gamma) \to L^2(\Gamma)$.

A classical solution is to use regularised versions of the combined formulation, composing the hypersingular operator with an operator of order -1, in other words, we precondition the hypersingular operator within the combined formulation. Several regularisers have been introduced, using approximation of pseudodifferential operators (see the recent review [4]), or integral operators and Calderón relations. We focused on the latter approach, where the regulariser is a single-layer operator associated with the complex frequency ik. This strategy was introduced in [2], and studied in [3]. In our work, we first derived new high-frequency estimates for these regularised formulations (see our preprint [5]), and then, frequency-explicit error bounds when using h-BEM to solve Neumann problems with these operators.

2 High-frequency estimates

We studied the integral operators

$$B_{k,\eta} := \mathrm{i}\eta \left(\frac{1}{2}I - K_k\right) + S_{\mathrm{i}k} H_k,$$

$$B'_{k,\eta} := \mathrm{i}\eta \left(\frac{1}{2}I - K'_k\right) + H_k S_{\mathrm{i}k},$$

where $\eta \in \mathbb{C} \setminus \{0\}$ and S_k, K_k, K'_k and H_k are the standard single-, double-, adjoint-double-layer and hypersingular operators defined for $k \in \mathbb{C}$. They both can be used to solve scattering problems with Neumann condition, $B'_{k,\eta}$ in the context of an indirect formulation, and $B_{k,\eta}$ for a direct formulation.

We proved new upper bounds on the norm of these operators. For example, when Γ is C^∞, we obtained

$$\|B_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} + \|B'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim |\eta|(1 + k^{1/4}\log(k+2)) + \log(k+2),$$

where $\lesssim$ means lower or equal up to a multiplicative constant independent of k and h. We also proved invertibility of these operators on $L^2(\Gamma)$, and upper bounds on the norm of the inverse, which is obtained expressing the inverse of these operators using the Dirichlet-to-Neumann map and the Impedance-to-Dirichlet map. The usual approach to derive bounds on the inverse of the integral operators is to use bounds on such maps. In the case of regularised formulations, the major difficulty stems from the fact that the Impedance-to-Dirichlet map is associated with the following non-standard problem:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ S_{\mathrm{i}k}\partial_n^- u - i\eta\gamma^- u = g & \text{on } \Gamma. \end{cases}$$

It led us to study this problem to finally obtain, for example when Γ is C^∞ and nontrapping:

$$\|(B_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} + \|(B'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{2/3},$$

where η is fixed.

3 Frequency-explicit error bounds

Equipped with the previous estimates, the next step was to find sufficient conditions on h under which we can obtain a frequency-explicit error bound. Note that, while results of this type have been derived for the combined-field formulation with Dirichlet boundary conditions, there was, to our knowledge, no such result for the Neumann problem.

Denoting V_h the approximation space, we assume the following approximation property $\min_{v_h \in V_h}\|u - v_h\|_{L^2(\Gamma)} \lesssim h^{p+1}\|u\|_{H^{p+1}(\Gamma)}$. We first proved sufficient conditions for quasioptimality: for k large enough, if

$$(hk)^{p+1}\,\mathrm{cond}(B_{k,\eta}) \lesssim 1,$$

then the Galerkin solution u_h exists, is unique, and satisfies the quasioptimality property $\|u - u_h\|_{L^2(\Gamma)} \lesssim \min_{v_h\in V_h}\|u - v_h\|_{L^2(\Gamma)}$. Then, using properties of the solution, we deduced that under the previous sufficient condition, we have the following error bound:

$$\frac{\|u - u_h\|_{L^2(\Gamma)}}{\|u\|_{L^2(\Gamma)}} \lesssim (\mathrm{cond}(B_{k,\eta}))^{-1},$$

with similar results holding for $B'_{k,\eta}$. This last result associated with the previous estimates gives a precise answer on how fast h needs to decrease to guarantee a frequency-explicit bound on the relative error.

References

[1] F. Ihlenburg and I. Babuška, Finite element solution of the Helmholtz equation with high wave number Part I: The h-version of the FEM. *Computers & Mathematics with Applications*, 1995.

[2] O. Bruno and T. Elling and C. Turc, Regularized integral equations and fast high-order solvers for sound-hard acoustic scattering problems. *International Journal for Numerical Methods in Engineering*, 2012.

[3] Y. Boubendir and C. Turc, Wave-number estimates for regularized combined field boundary integral operators in acoustic scattering problems with Neumann boundary conditions. *IMA Journal of Numerical Analysis*, 2013.

[4] X. Antoine and M. Darbas, An Introduction to Operator Preconditioning for the Fast Iterative Integral Equation Solution of Time-Harmonic Scattering Problems. *Multiscale Science and Engineering*, 2021.

[5] J. Galkowski and P. Marchand and E. Spence, High-frequency estimates on boundary integral operators for the Helmholtz exterior Neumann problem. *arXiv preprint: 2109.06017*, 2021