Scattering in a partially open waveguide: the forward problem

Laurent Bourgeois[1,*], **Sonia Fliss**[1], **Jean-François Fritsch**[2], **Christophe Hazard**[1], **Arnaud Recoquillay**[2]

[1]Laboratoire POEMS, ENSTA Paris, Palaiseau, France
[2]Université Paris-Saclay, CEA, List, F-91120, Palaiseau, France
*Email: laurent.bourgeois@ensta.fr

Abstract

We consider an acoustic scattering problem in a two-dimensional partially open waveguide, in the sense that the left part of the waveguide is closed, that is with a *bounded* cross-section, while the right part is bounded in the transverse direction by some Perfectly Matched Layers that mimic the situation of an open waveguide, that is with an *unbounded* cross-section. We prove well-posedness of such scattering problem, then propose and justify artificial conditions in the longitudinal direction based on Dirichlet-to-Neumann maps.

Keywords: Open waveguide, PMLs, DtN operators, Kondratiev theory

1 Introduction

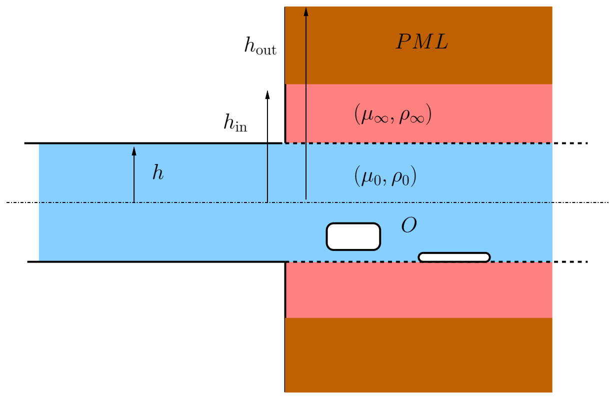
Let us introduce the domains $\Omega^- := (-\infty, 0) \times (-h, h)$, $\Omega^+ := (0, +\infty) \times (-h_{\text{out}}, h_{\text{out}})$ and $\Omega := \Omega^- \cup \Sigma_0 \cup \Omega^+$, with $\Sigma_0 := \{0\} \times (-h, h)$, as well as a bounded domain $O \subset (0, +\infty) \times (-h, h)$. We consider the problem: find u such that

$$\left\{ \begin{array}{rcll} Pu & = & 0 & \text{in } \Omega \setminus \overline{O}, \\ \partial_\nu u & = & 0 & \text{on } \partial\Omega, \\ u & = & 0 & \text{on } \partial O, \\ u - u^i & \text{is} & \text{outgoing}, & \end{array} \right. \tag{1}$$

where ν is the outward unit normal vector to $\partial\Omega$, u^i is a mode coming from the left waveguide, and the differential operator P is defined by

$$P := -\partial_y(\alpha\mu\, \partial_y \cdot) - \frac{\mu}{\alpha}\partial_{xx} \cdot - \frac{\mu}{\alpha}k^2,$$

with $(\alpha, \mu, k) := (1, \mu_0, k_0)$ in $\mathbb{R} \times (-h, h)$ (blue zone), $(\alpha, \mu, k) := (1, \mu_\infty, k_\infty)$ in $(0, +\infty) \times ((-h_{\text{in}}, -h) \cup (h, h_{\text{in}}))$ (pink zone) and $(\alpha, \mu, k) := (\alpha_\infty, \mu_\infty, k_\infty)$ in $(0, +\infty) \times ((-h_{\text{out}}, -h_{\text{in}}) \cup (h_{\text{in}}, h_{\text{out}}))$ (brown zone), with α_∞ a complex constant such that $\arg(\alpha_\infty) \in (-\frac{\pi}{2}, 0)$. We also assume that $k_0 < k_\infty$. The objective is to prove well-posedness of problem (1) in a appropriate functional space.



2 The uniform PML-waveguide

Let us consider the straight waveguide $\Omega_{\text{out}} := \mathbb{R} \times I_{\text{out}}$ with $I_{\text{out}} := (-h_{\text{out}}, h_{\text{out}})$, the coefficients of P being those of Ω^+: for a given f, find the outgoing solution u to the problem

$$\left\{ \begin{array}{rcll} Pu & = & f & \text{in } \Omega_{\text{out}}, \\ \partial_\nu u & = & 0 & \text{on } \partial\Omega_{\text{out}}. \end{array} \right. \tag{2}$$

The solutions in the form $u(x, y) = e^{\lambda x}\varphi(y)$ to problem (2) for $f = 0$ are called the modes. To specify them, we introduce the operator $\mathcal{L}(\lambda) : H^1(I_{\text{out}}) \to H^1(I_{\text{out}})^*$ defined by
$\langle \mathcal{L}(\lambda)\varphi, \psi \rangle_{H^1(I_{\text{out}})^*, H^1(I_{\text{out}})} :=$

$$\int_{I_{\text{out}}} (\alpha\mu d_y\varphi\, d_y\psi - \frac{\mu}{\alpha}(\lambda^2 + k^2)\varphi\, \psi)\, dy,$$

for all $\varphi, \psi \in H^1(I_{\text{out}})$. We denote by Λ the discrete set of $\lambda \in \mathbb{C}$ for which there exists a non zero eigenvector $\varphi \in H^1(I_{\text{out}})$ such that

$$\mathcal{L}(\lambda)\varphi = 0. \tag{3}$$

We assume that any eigenvector φ satisfies $\int_{I_{\text{out}}} \frac{\mu}{\alpha}\varphi^2\, dy \neq 0$, which guarantees that the λ_n are simple eigenvalues, both geometrically and algebraically. In addition, the fact that $k_0 < k_\infty$ enables us to write $\Lambda = \cup_{n=0}^{+\infty}\{-\lambda_n, \lambda_n\}$ with

$$\cdots \leq \Re e(\lambda_{n+1}) \leq \Re e(\lambda_n) \leq \cdots \leq \Re e(\lambda_0) < 0.$$

If the φ_n satisfy (3) for $\lambda = \pm\lambda_n$, up to a rescaling we have the biorthogonality condition

$\int_{I_{\text{out}}} \frac{\mu}{\alpha}\varphi_n\varphi_m\,dy = \delta_{mn}$. The modes (divided into leaky and PMLs modes) are given by $w_n^{\pm}(x,y) = e^{\pm\lambda_n x}\varphi_n(y)$ and are all evanescent. Since the transverse operator $\mathcal{L}$ is not self-adjoint, the φ_n do not form a complete orthonormal basis of $L^2(I_{\text{out}})$: the classical projection method used for a closed uniform waveguide fails. This is why we used the Kondratiev theory (see [1] for details and references) as the main ingredient of the proofs of both following theorems.
We introduce the space $\mathcal{W}^1_\beta(\Omega_{\text{out}})$ as the set of $v \in \mathcal{D}'(\Omega_{\text{out}})$ such that $e^{-\beta|x|}v$, $e^{-\beta|x|}\partial_x v$ and $e^{-\beta|x|}\partial_y v$ belong to $L^2(\Omega_{\text{out}})$.

Theorem 1 *For $f \in H^1(\Omega_{\text{out}})^*$, the problem (2) has a unique solution $u \in H^1(\Omega_{\text{out}})$. Furthermore, if $\beta > 0$ is such that Λ has no intersection with the line $\ell_{-\beta} := -\beta + i\mathbb{R}$ and $f \in \mathcal{W}^1_\beta(\Omega_{\text{out}})^*$, denoting $\Lambda \cap \{\lambda \in \mathbb{C},\ -\beta < \Re e\lambda < 0\} = \{\lambda_0, \lambda_1, \ldots, \lambda_{N_\beta-1}\}$, there exist some complex numbers a_n^+, a_n^- and a function $\tilde{u} \in \mathcal{W}^1_{-\beta}(\Omega_{\text{out}})$ such that*

$$u = \chi^+ \sum_{n=0}^{N_\beta-1} a_n^+ w_n^+ + \chi^- \sum_{n=0}^{N_\beta-1} a_n^- w_n^- + \tilde{u},$$

where $\chi^\pm \in C^\infty(\mathbb{R})$, $\chi^\pm = 1$ for $\pm x \geq 2L$ and $\chi^\pm = 0$ for $\pm x \leq L$.

Remark 2 *Since $k_0 < k_\infty$, the radiation condition simply consists here to search the solution in $H^1(\Omega_{\text{out}})$, which guarantees its uniqueness.*

3 The scattering problem of interest

In order to analyze problem (1), we formulate an equivalent one set in the bounded domain D_M, that is the domain $\Omega \setminus \overline{O}$ truncated by the sections Σ_0 and $\Sigma_M := \{M\} \times (-h_{\text{out}}, h_{\text{out}})$. We introduce the DtN operator

$$\begin{aligned} T_0 :\ H^{1/2}(\Sigma_0) &\to\ H^{1/2}(\Sigma_0)^* \\ \varphi &\mapsto\ T_0\varphi = -\mu_0\partial_x u^-|_{\Sigma_0}, \end{aligned}$$

where u^- is the solution to the left half-waveguide problem with Dirichlet data φ on Σ_0, and the DtN operator with an overlap $(M-L)$

$$\begin{aligned} T_{L,M} : H^{1/2}(\Sigma_L) &\to\ H^{1/2}(\Sigma_M)^* \\ \varphi &\mapsto\ T_{L,M}\varphi = \frac{\mu}{\alpha}\partial_x u^+|_{\Sigma_M}, \end{aligned}$$

where u^+ is the solution to the right half-waveguide problem with Dirichlet data φ on Σ_L, with $L \leq M$. That the operator T_0 is well-defined and has the form of a series is classical. Using the first part of the previous theorem and a symmetry argument, we prove that $T_{L,M}$ is well-defined as well. Then we introduce the problem set in D_M: find $u \in H^1(D_M)$ such that

$$\left\{\begin{array}{rcll} Pu &=& 0 & \text{in } D_M, \\ \partial_\nu u &=& 0 & \text{on } \Gamma_M, \\ u &=& 0 & \text{on } \partial O, \\ -\mu_0\partial_x u &=& T_0(u|_{\Sigma_0}) - 2\mu_0\partial_x u^i & \text{on } \Sigma_0, \\ \dfrac{\mu}{\alpha}\partial_x u &=& T_{L,M}(u|_{\Sigma_L}) & \text{on } \Sigma_M. \end{array}\right. \tag{4}$$

with $\Gamma_M = \partial D_M \setminus (\Sigma_0 \cup \Sigma_M \cup \partial O)$. Having in mind the numerical computation of the solution, we also introduce an operator which approximates the DtN operator $T_{L,M}$ with the help of the eigenvalues and eigenfunctions (λ_n, φ_n) of the transverse operator $\mathcal{L}$, that is for a fixed $\beta > 0$, $T_{L,M}(\varphi)$ is replaced for $\varphi \in H^{1/2}(\Sigma_L)$ by

$$T^\beta_{L,M}(\varphi) = \sum_{n=0}^{N_\beta-1} \frac{\mu}{\alpha}\lambda_n e^{\lambda_n(M-L)} \left(\int_{\Sigma_L} \frac{\mu}{\alpha}\varphi\varphi_n\,dy\right)\varphi_n.$$

Theorem 3 *Problems (1) and (4) are equivalent. Problem (4) has a unique solution u in $H^1(D_M)$ iff the same problem for $u^i = 0$ has only the trivial solution. Moreover, using the same assumption on β and the same notation as in Theorem 1, there exist some complex numbers a_n^+ and a function $\tilde{u} \in \mathcal{W}^1_{-\beta}(\Omega^+ \setminus \overline{O})$ such that $\tilde{u}|_{\partial O} = 0$ satisfying in $\Omega^+ \setminus \overline{O}$*

$$u = \chi^+ \sum_{n=0}^{N_\beta-1} a_n^+ w_n^+ + \tilde{u}.$$

For a fixed L and M large enough, if problem (1) is well-posed, then the problem (4) with operator $T_{L,M}$ replaced by $T^\beta_{L,M}$ is well-posed as well. Moreover, denoting $u^\beta_{L,M}$ the corresponding solution, there exists a constant $C_\beta > 0$ which is independent of M such that

$$\|u - u^\beta_{L,M}\|_{H^1(D_L)} \leq C_\beta\, e^{-\beta(M-L)}\|u\|_{H^1(\Omega^+\setminus\overline{O})}.$$

References

[1] L. Bourgeois, S. Fliss, J.-F. Fritsch, C. Hazard and A. Recoquillay, Scattering in a partially open waveguide: the forward problem, *submitted.*