Scattering in a partially open waveguide: the inverse problem

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Abstract

Guided waves can be used to perform nondestructive testing of partially buried elongated structures present in various industrial sectors such as civil engineering or oil and gas. Such an inverse problem presents two main challenges: we can access to only one side of the zone to control, and the waves propagating in the structure partially leak in the surrounding medium. The buried part of the structure is truncated in the transverse direction with Perfectly Matched Layers (PMLs), which enable us to treat the domain as the junction of two closed waveguides with complex coefficients. The Linear Sampling Method (LSM) is adapted to this new configuration.

Keywords: Inverse scattering, Linear Sampling Method, buried waveguide, PML

1 Intoduction

The closed part of the structure occupies the domain $\Omega^- := \mathbb{R}^- \times (-h,h)$ with $h>0$ and its buried part the domain $\Omega^+_\infty := \mathbb{R}^+ \times \mathbb{R}$. The shear modulus and the density are piecewise positive constants given by $(\mu,\rho) = (\mu_0,\rho_0)$ in $\mathbb{R}\times(-h,h)$ and $(\mu,\rho) = (\mu_\infty,\rho_\infty)$ in $\mathbb{R}^+ \times \big((-\infty,h)\cup(h,+\infty)\big)$. The speed and the wavenumber are respectively given by $c := \sqrt{\mu/\rho}$ and $k := \omega/c$ where ω is the angular frequency. The defect we want to identify occupies a smooth and bounded domain O which lies either in the core $\mathbb{R}^+\times(-h,h)$ of the buried part, or in its sheath $\mathbb{R}^+ \times \big((-h_{in},h)\cup(h,h_{in})\big)$ with $h_{in} > h$. The domain Ω^+_∞ is truncated in the transverse direction with a bounded PML occupying the domain $\mathbb{R}^+ \times \big((-h_{out},h_{in})\cup(h_{in},h_{out})\big)$ for $h_{out} > h_{in}$; the truncated domain is denoted by $\Omega^+ := \mathbb{R}^+ \times (-h_{out},h_{out})$. We also define $\Omega := \Omega^- \cup \Sigma_0 \cup \Omega^+$ with $\Sigma_0 := \{0\}\times(-h,h)$. A point in Ω is denoted by $\mathbf{x} = (x,y) \in \mathbb{R}^2$, and the PML function is given by

$$\alpha(x,y) := \begin{cases} 1 & |y| \le h \\ \alpha_\infty & |y| > h, \end{cases}$$

with $-\frac{\pi}{2} < \arg(\alpha_\infty) < 0$. In order to write the forward problem, we introduce the modes of the left half-guide, which are the functions $\tilde{w}_n^\pm(x,y) = e^{\pm\tilde{\lambda}_n x}\tilde{\varphi}_n(y)$, with $(\tilde{\lambda}_n,\tilde{\varphi}_n)$ satisfying

$$\begin{cases} -d_y^2\tilde{\varphi}_n - k_0^2\tilde{\varphi}_n = \tilde{\lambda}_n^2\tilde{\varphi}_n & \text{in } (-h,h) \\ d_y\tilde{\varphi}_n = 0 & \text{on } y = \pm h. \end{cases}$$

The scattering problem reads:
find $\varphi \in H^1_{loc}(\Omega\backslash\bar{O})$ *such that*

$$\begin{cases} -\Delta\varphi - k_0^2\varphi = 0 & \text{in } \Omega^- \\ -\frac{\alpha}{\mu}\partial_y(\alpha\mu\partial_y\varphi) - \partial_x^2\varphi - k^2\varphi = 0 & \text{in } \Omega^+\backslash\bar{O} \\ [\![\varphi]\!] = 0 & \text{on } \Sigma_0 \\ [\![\partial_x\varphi]\!] = 0 & \text{on } \Sigma_0 \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } \partial\Omega \\ \varphi = 0 & \text{on } \partial O \\ \varphi - \tilde{w}^+_{n,0} & \text{is outgoing,} \end{cases} \tag{1}$$

where $\tilde{w}^+_{n,0}$ is the extension of the mode $\tilde{w}^+_n$ of the left half-guide to the positive x by 0 and $\mathbf{n}$ is the outgoing unit normal to Ω. In order to identify the obstacle O, we use the fact that our domain consists of the junction of two closed waveguides. In the following, we present first how the LSM can be adapted to the junction of two closed waveguides with real parameters. Then, we explain how this work has been extended to our configuration of interest, underlying the similarities but also differences arising with the PMLs.

2 The LSM for a junction of closed waveguides with real coefficients

We summarize here the main results obtained in [1] for a junction of an arbitrary number of waveguides, but limit the presentation to the simpler case of back-scattering identification in a junction of two branches with a constant wavenumber. To begin with, we define the reference fields $\tilde{r}_n$ which are the solution of the problems:
find $\tilde{r}_n \in H^1_{loc}(\Omega)$ *such that*

$$\begin{cases} -\Delta\tilde{r}_n - k^2\tilde{r}_n = 0 & \text{in } \Omega \\ \partial_{\mathbf{n}}\tilde{r}_n = 0 & \text{on } \partial\Omega \\ \tilde{r}_n - \tilde{w}^+_{n,0} & \text{is outgoing.} \end{cases} \tag{2}$$

Then, we introduce for $\mathbf{x}' \in \Omega$ the Green function solution of: *find* $\mathcal{G}(\cdot, \mathbf{x}') \in L^2_{loc}(\Omega)$ *such that*

$$\begin{cases} -\Delta\mathcal{G}(\cdot, \mathbf{x}') - k^2\mathcal{G}(\cdot, \mathbf{x}') = \delta_{\mathbf{x}'} & \text{in } \Omega \\ \partial_{\mathbf{n}}\mathcal{G}(\cdot, \mathbf{x}') = 0 & \text{on } \partial\Omega \\ \mathcal{G}(\cdot, \mathbf{x}') & \text{is outgoing.} \end{cases} \tag{3}$$

Theorem 1 *Problems* (2) *and* (3) *are well-posed, and we have for* $x' < 0$ *and* $x \geq x'$ *the decomposition:*

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = -\sum_{n\geq 0} \frac{1}{2\tilde{\lambda}_n} \tilde{r}_n(\mathbf{x}')\tilde{w}_n^-(\mathbf{x}). \tag{4}$$

The decomposition (4) is obtained using the reciprocity property of the Green function. For $F \in H^{\frac{1}{2}}(\partial O)$, we also introduce the scattering problem: *find* $u^s \in H^1_{loc}(\Omega\backslash\bar{O})$ *such that*

$$\begin{cases} -\Delta u^s - k^2 u^s = 0 & \text{in } \Omega\backslash\bar{O} \\ \partial_{\mathbf{n}} u^s = 0 & \text{on } \partial\Omega \\ u^s = F & \text{on } \partial O \\ u^s & \text{is outgoing.} \end{cases} \tag{5}$$

The inverse problem consists in finding O from the measurement of the u^s_n on $\Sigma_{-R} := \{-R\} \times (-h, h)$, $R > 0$, for all $n \in \mathbb{N}$, where u^s_n is the solution of (5) for $F = -\tilde{r}_{n|\partial O}$. Let $\mathbf{z} = (x_z, y_z)$, $x_z \geq -R$ be a sampling point. Let us introduce the so-called near-field equation: *find* $h \in L^2(\Sigma_{-R})$ *such that*

$$\int_{\Sigma_{-R}} u^s(\mathbf{x}, \mathbf{x}')h(\mathbf{x}')d\mathbf{x}' = \mathcal{G}(\mathbf{x}, \mathbf{z}),\ \forall \mathbf{x} \in \Sigma_{-R}, \tag{6}$$

where $u^s(\cdot, \mathbf{x}')$ is the solution of problem (5) with $F = -\mathcal{G}(\cdot, \mathbf{x}')_{|\partial O}$. Using (4), we derive the following modal formulation of the LSM.

Proposition 2 *The near-field equation* (6) *reads: find* $h(\mathbf{z}) = \sum_{n\geq 0} h_n(\mathbf{z})\tilde{\varphi}_n \in L^2(\Sigma_{-R})$ *such that*

$$\sum_{n\geq 0} \frac{e^{\tilde{\lambda}_n R}}{\tilde{\lambda}_n} U_{mn} h_n(\mathbf{z}) = \frac{e^{\tilde{\lambda}_n R}}{\tilde{\lambda}_n} \tilde{r}_n(\mathbf{z}),\ \forall m \in \mathbb{N}, \tag{7}$$

with $u^s_{n|\Sigma_{-R}} = \sum_{m\geq 0} U_{mn}\tilde{\varphi}_n$.

To image the defect, we plot the norm of the solution $h^{reg}(\cdot)$ of a Tikhonov-Morozov regularization of system (7) with the series truncated to the propagating modes, which are the modes satisfying $\tilde{\lambda}_n \in i\mathbb{R}$. It is important to stress that the reference fields are computed once and for all during an offline phase, which leads to an efficient sampling procedure.

3 The LSM in a partially buried waveguide with a transverse PML

A partially buried waveguide truncated in its embedded part with transverse PMLs can be viewed as the junction of two closed waveguides, one of which involves a complex parameter. The LSM can be then derived like before by introducing the reference fields, the Green function and the near-field equation of the new structure. As for its mathematical justification, it presents two main difficulties: proving the well-posedness of problem (1) and obtaining a rigorous decomposition of the Green function on the reference fields. The first difficulty was adressed in [2], and the second one in a forthcoming article. These difficulties are consequences of the loss of the self-adjointness of the transverse operator appearing in the mathematical analysis, which is a consequence of the introduction of the complex-valued function α modelling the PML. The considered materials satisfy $c_0 > c_\infty$, which implies that there are no propagating modes in the right half-guide. Thus, all the waves leak in the surrounding medium, which results in a loss of information for the imaging. A numerical result is finally presented with Figure 1.

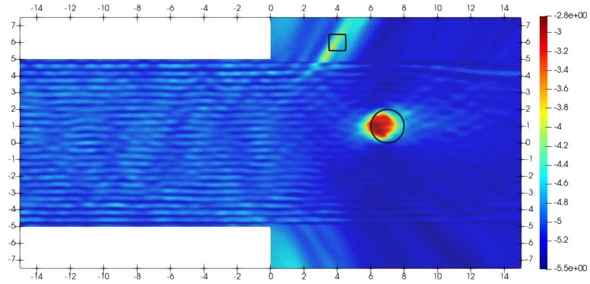

Figure 1: Level curves of the imaging function $\mathbf{z} \mapsto -\log(\|h^{reg}(\mathbf{z})\|_{L^2(\Sigma_{-R})})$ for two Dirichlet obstacles in a steel plate partially buried in concrete, using 26 propagating modes. Parameters: $h = 5$, $h_{in} = 7.5$, $h_{out} = 12.5$, 10% of noise.

References

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