Computing eigenvalues of the Laplacian on rough domains

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Abstract

We prove sufficient conditions under which the eigenvalues of a sequence of Dirichlet Laplace operators on bounded domains converge to the eigenvalues of a given limit domain. Our hypotheses allow for a wide variety of domains with cusps and fractal boundaries. This result is applied to prove the convergence of a numerical algorithm for computing the eigenvalues of such domains.

Keywords: Mosco convergence, spectral approximation, rough boundaries.

1 Introduction

This short paper focuses on the main technical results of our paper [1], where we study the *Solvability Complexity Indices (SCI)* [2] of computational eigenvalue problems associated to the Dirichlet Laplacian on bounded domains.

Throughout this paper, we consider a sequence of bounded domains $\mathcal{O}_n \subset \mathbb{R}^2$, $n \in \mathbb{N}$, converging to a bounded domain $\mathcal{O} \subset \mathbb{R}^2$ in the sense that

$$l(n) := \mathrm{dist}_H(\mathcal{O}_n, \mathcal{O}) + \mathrm{dist}_H(\partial\mathcal{O}_n, \partial\mathcal{O}) \to 0$$

as $n \to \infty$, where dist_H denotes the Hausdorff distance. Let $-\Delta_{\mathcal{O}_n}$ and $-\Delta_{\mathcal{O}}$ denote the corresponding Laplace operators, endowed with Dirichlet boundary conditions. We denote the eigenvalues of $-\Delta_{\mathcal{O}_n}$ by $\lambda_k(\mathcal{O}_n)$, $k \in \mathbb{N}$, and the eigenvalues of $-\Delta_{\mathcal{O}}$ by $\lambda_k(\mathcal{O})$, $k \in \mathbb{N}$.

Our main result Theorem 3 concerns *Mosco convergence*. It implies the following result concerning the convergence of eigenvalues.

Theorem 1 *Suppose that $\partial\mathcal{O}$ is a Jordan curve with zero Lebesgue measure and that, for each $n \in \mathbb{N}$, $\partial\mathcal{O}_n$ is locally connected. Then, for each $k \in \mathbb{N}$, we have*

$$\lambda_k(\mathcal{O}_n) \to \lambda_k(\mathcal{O}) \quad as \quad n \to \infty.$$

In fact, this result is proved in [1] under more general hypotheses which allow for multiple boundary components. Note that a Jordan curve need not be locally the map of a continuous function, may contain cusps and may have Haussdorff or upper box-counting dimension of up to two.

2 Mosco convergence

The notion of Mosco convergence can be stated for general Banach spaces, however, here we restrict our attention to H_0^1 Sobolev spaces.

Definition 2 *We have convergence in the Mosco sense $H_0^1(\mathcal{O}_n) \xrightarrow{M} H_0^1(\mathcal{O})$ as $n \to \infty$ if*

(i) For all $u \in H_0^1(\mathcal{O})$, there exists $u_n \in H_0^1(\mathcal{O}_n)$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \|u_n - u\|_{H^1} = 0$.

(ii) For any subsequence $H_0^1(\mathcal{O}_{n_j})$, $j \in \mathbb{N}$, and any $u_j \in H_0^1(\mathcal{O}_{n_j})$, $j \in \mathbb{N}$, such that $u_j \rightharpoonup u$ as $j \to \infty$ in H^1 for some $u \in H^1(\mathbb{R}^2)$, we have $u \in H_0^1(\mathcal{O})$.

It is well known that, since the domains $\mathcal{O}_n$ and $\mathcal{O}$ are bounded, $H_0^1(\mathcal{O}_n) \xrightarrow{M} H_0^1(\mathcal{O})$ implies the convergence for the eigenvalues of the Dirichlet Laplace operators. Hence, the next result implies Theorem 1.

Theorem 3 *Under the hypotheses of Thm. 1, we have*

$$H_0^1(\mathcal{O}_n) \xrightarrow{M} H_0^1(\mathcal{O}) \quad as \quad n \to \infty. \tag{1}$$

Let us now outline the main steps of the proof, assuming all stated hypotheses.

A) From uniform Poincaré to Mosco

Firstly, we are able to reduce Mosco convergence to the verification of certain Poincaré-type inequalities on neighbourhoods of the boundary of the form

$$\partial^r\mathcal{O} := \{x \in \mathcal{O} : \mathrm{dist}(x, \partial\mathcal{O}) < r\}, \quad r > 0.$$

Proposition 4 *Suppose that there exists a sequence $\epsilon(n) \geq 2l(n)$, $n \in \mathbb{N}$, with $\epsilon(n) \to 0$ as $n \to \infty$ and constants $C, \alpha > 0$ such that*

$$\|u\|_{L^2(\partial^{\epsilon(n)}\mathcal{O})} \leq C\epsilon(n)\|\nabla u\|_{L^2(\partial^{\alpha\epsilon(n)}\mathcal{O})}, \tag{2}$$

$$\|v\|_{L^2(\partial^{\epsilon(n)}\mathcal{O}_n)} \leq C\epsilon(n)\|\nabla v\|_{L^2(\partial^{\alpha\epsilon(n)}\mathcal{O}_n)} \tag{3}$$

for all $n \in \mathbb{N}$, $u \in H_0^1(\mathcal{O})$ and $v \in H_0^1(\mathcal{O}_n)$. Then, (1) holds.

It therefore suffices to prove (2) and (3).

B) Poincaré for a single domain

Inequality (2) follows from the following result.

Theorem 5 *For $r \in (0, r_0)$ and $u \in H_0^1(\mathcal{O})$,*

$$\|u\|_{L^2(\partial^r \mathcal{O})} \leq 5r \|\nabla u\|_{L^2(\partial^{2\sqrt{2}r}\mathcal{O})}$$

where $r_0 = (4\sqrt{2})^{-1} Q(\partial\mathcal{O})$ and

$$Q(\partial\mathcal{O}) := \inf \left\{ \operatorname{diam}(\Gamma) : \Gamma \subset \partial\mathcal{O} \begin{array}{c}\text{path}\\ \text{component}\end{array} \right\}.$$

Here, the path components refer to the equivalence classes under the relation $x \sim y \iff x$ connected by a path to y. In fact, Theorem 5 holds for an arbitrary open set $\mathcal{O} \subset \mathbb{R}^2$ with $Q(\partial\mathcal{O}) > 0$. We prove this result by a geometric method involving explicitly constructing a certain type of bundle of paths from points in $\partial^r\mathcal{O}$ to the boundary $\partial\mathcal{O}$.

C) Poincaré for a sequence of domains

Under our hypotheses, we can provide a geometric description of $\partial\mathcal{O}_n$ for large n. More precisely, we show that there exists a sequence $\epsilon(n)$ as in Proposition 4 such that, for all large enough n, $\partial\mathcal{O}_n$ has a path-connected subset Γ_n whose diameter exceeds $\operatorname{diam}(\partial\mathcal{O}) - \epsilon(n)$ and such that any other point in $\partial\mathcal{O}_n$ lies within a distance $\epsilon(n)$ to Γ_n.

As it turns out, by applying Theorem 5 to the domain $\mathcal{V}_n = \Gamma_n^c$, we are able to verify inequality (3) with $C = 10$ and $\alpha = 4\sqrt{2}$.

3 Pixelated domain algorithm

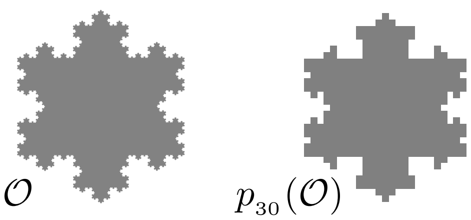

Figure 1: A pixelated domain approximation for the interior of a Koch snowflake.

Consider any domain $\mathcal{O}$ satisfying the hypotheses of Theorem 1. Suppose we have access to the information of whether or not a given point $x \in \mathbb{R}^2$ is in $\mathcal{O}$. Using the above results, we are able to construct a simple numerical method for the eigenvalues of $-\Delta_{\mathcal{O}}$.

This numerical method is based on *pixelated domain approximations*, which are defined by

$$p_n(\mathcal{O}) := \operatorname{int}\left(\bigcup_{j \in L_n(\mathcal{O})} \left(j + [-\tfrac{1}{2n}, \tfrac{1}{2n}]^2\right) \right)$$

where

$$L_n(\mathcal{O}) := \left\{ j \in (n^{-1}\mathbb{Z})^2 : j \in \mathcal{O} \right\}.$$

Under the stated hypotheses for $\mathcal{O}$, we are able to prove that pixelated domains converge in the sense that

$$\operatorname{dist}_H(p_n(\mathcal{O}), \mathcal{O}) + \operatorname{dist}_H(\partial p_n(\mathcal{O}), \partial\mathcal{O}) \to 0$$

as $n \to \infty$. Consequently, Theorem 1 guarantees that the eigenvalues of $-\Delta_{p_n(\mathcal{O})}$ converge to the eigenvalues of $-\Delta_{\mathcal{O}}$.

Due to their regular shape, pixelated domains may be easily triangulated, for instance with a uniform mesh. Hence the eigenvalues of $-\Delta_{p_n(\mathcal{O})}$ may be approximated using a finite element scheme, in turn providing an approximation for the eigenvalues of $-\Delta_{\mathcal{O}}$.

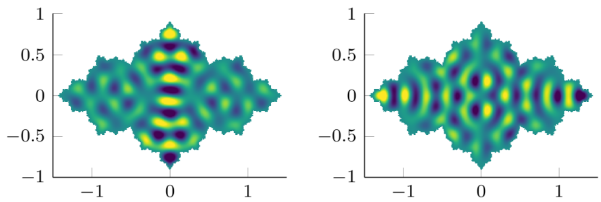

Figure 2: A pixelation and FEM approximation for the 95th and 99th eigefunction of the Dirichlet Laplacian $-\Delta_{\mathrm{int}K(f_c)}$ with $c = \frac{\sqrt{5}-1}{2}$.

As an application of our results we are able to approximate the eigenvalues of a family of *filled Julia sets* $K(f_c)$.

Example 6 *Let $f_c(z) = z^2 + c$, where $|c| < \frac{1}{4}$, and consider the compact set*

$$K(f_c) := \{z_0 \in \mathbb{C} : (f^{\circ n}(z_0))_{n \in \mathbb{N}} \text{ bounded}\}.$$

Then, the bounded domain $\operatorname{int}(K(f_c))$ satisfies the hypotheses of Thm. 1.

References

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