Perfectly Matched Layers for second order Maxwell's equations in time domain

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Abstract

In this work, we are interested in solving second order time-domain Maxwell's equations in an unbounded domain. To restrict the computational domain, we propose a Perfectly Matched Layers (PMLs) formulation which implies only a reduced number of auxiliary functions (thus reducing the computational cost). This formulation has been validated numerically.

Keywords: Perfectly Matched Layers, Maxwell's equations, time-domain.

An introduction

Maxwell's equations appears in many contexts, especially in non-invasive procedures (in biomedical applications, geophysics, items detection, structures control...). In general, an unbounded domain is used to model the interaction between electromagnetic waves and the perturbations (such as tumor, rocks, buried item or water infiltration). One way to restrict the computational domain is to use PMLs, which consists in surrounding the bounded physical domain of interest by artificial non-reflecting absorbing layers. They have been initially developed for first order formulation of Maxwell's equations (magnetic and electric field). More recently, the PMLs interpretation as a complex change for variables derived a new formulation for Maxwell's equations in second order. Yet, although this approach has been well-developed for scalar wave equation (or TMz EM field [1]), it has not been reported to the best of our knowledge for the full electrical field.

In this work, we propose a PML formulation for second order time-domain Maxwell's equations in dimension 2. Surprisingly, it leads to the introduction of "only" two auxiliary functions, as in TMz formulation.

1 In time-harmonic regime

The results are given for the two dimensional case, the three dimensional case will be discuss during the talk. Considering no magnetic source and constitutive law for linear materials, the Maxwell's equations read

$$\underline{\nabla} \times \mu^{-1} \nabla \times \underline{\mathcal{E}} + \partial^2_{tt} \varepsilon \underline{\mathcal{E}} = \underline{\mathcal{S}}.$$

In time-harmonic domain, the equations become

$$\underline{\nabla} \times \mu^{-1} \nabla \times \underline{E} - \omega^2 \varepsilon \underline{E} = \underline{S}, \tag{1}$$

where $\underline{E}$ (resp. $\underline{S}$) is the Fourier transform in time of $\underline{\mathcal{E}}$ (resp. $\underline{\mathcal{S}}$). In time-harmonic regime, the PML formulation consists in applying a complex stretching $x \to L_x(x)x$ and $y \to L_y(y)y$, where here we take $L_t(t) = 1 - \mathrm{i}\frac{d(t)}{\omega}$, $d(t) = \left(\frac{l \pm (t - t_\Gamma)}{l}\right)^2$, t_Γ being the boundary in the t direction and l the PMLs width. This change of variables amounts in replacing in equation (1) the derivative with respect to x and y by $\frac{1}{L_x}\partial_x$ and $\frac{1}{L_y}\partial_y$. Applying PMLs, we get

$$\underline{\nabla} \times \mu^{-1} \nabla_L \times \underline{\tilde{E}} - \omega^2 \varepsilon L_x L_y \, \underline{\tilde{E}} = \underline{S}_e, \tag{2}$$

where:

- $\underline{\tilde{E}} = \begin{bmatrix} L_x^{-1} & 0 \\ 0 & L_y^{-1} \end{bmatrix} \underline{E}$,
- $\nabla_L = \begin{bmatrix} L_y L_x^{-1} \partial_x & L_x L_y^{-1} \partial_y \end{bmatrix}$,

and the 2D curl operators are defined by:

$$\underline{\nabla} \times f = \begin{pmatrix} \partial_y f \\ -\partial_x f \end{pmatrix}, \ \nabla \times \underline{u} = \partial_x u_2 - \partial_y u_1.$$

Note that, we assume that the support of the source term is not in the PMLs, so it will not be impacted by the coefficients L_x and L_y.

2 Back to time-domain regime

Very classically, coming back to time-domain involves time-convolution for the terms multiplied

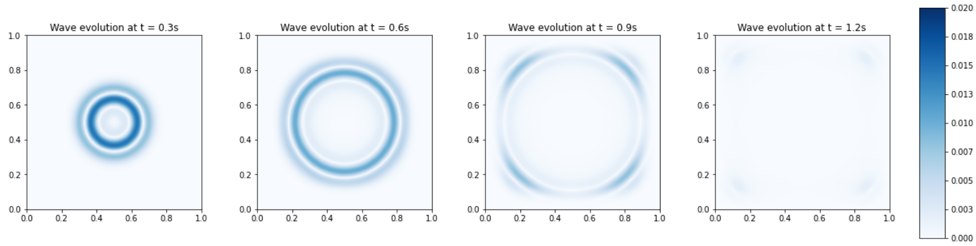

Figure 1: Magnitude of $\underline{\mathcal{E}}$ at different times.

by PMLs coefficients. To avoid these convolution, the idea is to introduce auxiliary functions. First, we have:

$$\mathcal{F}^{-1}\left(-\omega^2 L_x L_y \underline{\tilde{E}}\right),$$

which gives:

$$\partial_{tt}^2\underline{\tilde{\mathcal{E}}} + (d(x)+d(y))\,\partial_t\underline{\tilde{\mathcal{E}}} + d(x)d(y)\partial_t\underline{\tilde{\mathcal{E}}}. \quad (3)$$

Second, we set:

$$\mathcal{F}^{-1}\left(\left[-L_xL_y^{-1}\partial_y \quad L_yL_x^{-1}\partial_x\right]\underline{\tilde{E}}\right).$$

Let $C_x = -L_xL_y^{-1}\partial_y\tilde{E}_x$ and $C_y = L_yL_x^{-1}\partial_x\tilde{E}_y$, implying

$$\left|\begin{array}{l} \partial_t(-\partial_y\tilde{\mathcal{E}}_x - \mathcal{C}_x) - d(x)\partial_y\tilde{\mathcal{E}}_x + d(y)\mathcal{C}_x = 0, \\ \partial_t(\partial_x\tilde{\mathcal{E}}_y - \mathcal{C}_y) + d(y)\partial_x\tilde{\mathcal{E}}_y - d(x)\mathcal{C}_y = 0, \end{array}\right.$$

where $\mathcal{C}_x$ and $\mathcal{C}_y$ are the inverse Fourier transform of C_x and C_y. Then, setting $\mathcal{P}_x = -\partial_y\tilde{\mathcal{E}}_x - \mathcal{C}_x$ and $\mathcal{P}_y = \partial_x\tilde{\mathcal{E}}_y - \mathcal{C}_y$, we get:

$$\left|\begin{array}{l} \partial_t\mathcal{P}_x - (d(x)-d(y))\,\partial_y\tilde{\mathcal{E}}_x + d(y)\mathcal{P}_x = 0, \\ \partial_t\mathcal{P}_y - (d(y)-d(x))\,\partial_x\tilde{\mathcal{E}}_y + d(x)\mathcal{P}_y = 0, \end{array}\right.$$

which finally gives

$$\mathcal{F}^{-1}(\mathcal{C}_x + \mathcal{C}_y) = \nabla \times \underline{\tilde{\mathcal{E}}} - (\mathcal{P}_x + \mathcal{P}_y). \quad (4)$$

To sum up, the new formulation in time-domain is:

$$\partial_{tt}^2\underline{\tilde{\mathcal{E}}} + (d(x)+d(y))\,\partial_t\underline{\tilde{\mathcal{E}}} + d(x)d(y)\underline{\tilde{\mathcal{E}}} + \underline{\nabla} \times \mu^{-1}\nabla \times \underline{\tilde{\mathcal{E}}} - \underline{\nabla} \times (\mathcal{P}_x + \mathcal{P}_y) = \underline{\mathcal{S}}, \quad (5)$$

with the equation on $\underline{\mathcal{P}}$:

$$\partial_t\underline{\mathcal{P}} - (d(x)-d(y))\begin{bmatrix} \partial_y\tilde{\mathcal{E}}_x \\ \partial_x\tilde{\mathcal{E}}_y \end{bmatrix} + \underline{\underline{\mathcal{D}}}\,\underline{\mathcal{P}} = \underline{0}, \quad (6)$$

with $\underline{\underline{\mathcal{D}}} = \begin{bmatrix} d(y) & 0 \\ 0 & d(x) \end{bmatrix}.$

3 Numerical results

The stable discretization of such system (5-6) is not obvious, see [2]. On Figure 1, the solution has been computed with a Finite Differences scheme that will be detailed during the talk. On Figure 3, we have represented the evolution in time of the energy. Further works are currently in progress to implement Finite Elements discretization in two and three dimensions.

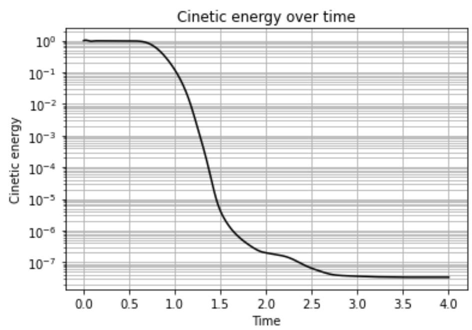

Figure 2: Calculation of the energy dissipation over time

References

[1] Björn Sögreen a, N. Anders Petersson, Perfectly matched layers for Maxwell's equations in second order formulation, *Journal of Computational Physics* **209** (2005), pp. 19–46.

[2] D.H. Baffet, M. Grote, S. Imperiale and M. Kachanovska, Energy decay and stability of a perfectly matched layer for the wave equation, *Journal of Scientific Computing* **81(3)** (2019), pp. 2237–117.