# A phase-field approach to shape and topology optimization of nonlinear acoustic waves

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**Abstract**

We investigate the problem of finding the optimal shape and topology of a system of acoustic lenses in a dissipative medium, where the sound propagation is governed by a general semilinear strongly damped wave equation. We introduce a phase-field formulation of this problem through diffuse interfaces between the lenses and the surrounding fluid. The resulting formulation is shown to be well-posed and we rigorously derive first-order optimality conditions for this problem. Additionally, we establish a relation between the diffuse interface problem and a perimeter-regularized sharp interface shape optimization problem via the $\Gamma$-limit of the reduced objective. The talk is based on [1].

**Keywords:** shape and topology optimization, nonlinear acoustics, phase-field methods, optimality conditions, $\Gamma$-convergence

## 1 Introduction

We consider an acoustic lens system in a thermoviscous fluid. A number of acoustic lenses $\Omega_{l,1}, \dots, \Omega_{l,n}$ of the same material are immersed in an acoustic fluid $\Omega_f$, $n \in \mathbb{N}$; see Figure 1. The material parameters corresponding to the lens are given by $(c_l, b_l, k_l)$ and to the fluid by $(c_f, b_f, k_f)$. Here $c_i > 0$ is the speed of sound, $b_i > 0$ the sound diffusivity, and $k_i \in \mathbb{R}$ is the nonlinearity coefficient, where $i \in \{l, f\}$.

The goal is to determine the number and shape of acoustic lenses so that we reach the desired pressure distribution $u_\mathrm{d} \in L^2(0,T; L^2(\Omega))$ in some region of interest $D \subset \Omega$, where $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a hold-all domain, assumed to be Lipschitz regular. Let $T > 0$ denote the final time of propagation. Assuming that we have a high-intensity or high-frequency sound source, the propagation of sound waves is nonlinear. We can obtain the pressure field $u$ by solving

$$\alpha(x,t)u_{tt} - \operatorname{div}(c^2\nabla u) - \operatorname{div}(b\nabla u_t) = f(u_t)$$

on $\Omega\times(0,T)$, with the right-hand side nonlinearity given by $f(u_t) = 2ku_t^2$. The medium parameters are piecewise constant functions, defined as

$$\begin{aligned} c &= c_l\chi_{\Omega_l} + c_f(1-\chi_{\Omega_l}), \\ b &= b_l\chi_{\Omega_l} + b_f(1-\chi_{\Omega_l}), \\ k &= k_l\chi_{\Omega_l} + k_f(1-\chi_{\Omega_l}), \end{aligned} \tag{1}$$

with $\Omega_l = \bigcup_{j=1}^{n} \Omega_{l,j}$. We assume that the coefficient $\alpha$ does not degenerate, that is, we assume that there exist $\underline{\alpha}$, $\overline{\alpha} > 0$, such that

$$\underline{\alpha} \leqslant \alpha(x,t) \leqslant \overline{\alpha} \qquad \text{a.e. in } \Omega \times (0,T). \tag{2}$$

Equation (1) can be seen as a semi-linearization of the Westervelt equation obtained by freezing the term $\alpha(u) = 1 - 2ku$. The sound waves are excited via boundary in form of Neumann boundary conditions

$$c^2\frac{\partial u}{\partial n} + b\frac{\partial u_t}{\partial n} = g \quad \text{on} \quad \Gamma = \partial\Omega, \tag{3}$$

where $n$ denotes the unit outward normal to $\Gamma$ and the problem is additionally supplemented with initial conditions.

## 2 A phase-field approach

We next introduce a continuous material representation between lenses and fluid by employing diffuse interfaces $\xi_i$, $i \in [1,n]$, with thickness proportional to $\varepsilon > 0$. We define a partition

$$\Omega = \Omega_f \cup \overline{\xi} \cup \Omega_l$$

of $\Omega$, where $\xi = \bigcup_{i=1}^n \xi_i$ and then also introduce a phase-field function $\varphi$, such that

$$\begin{aligned} \varphi(x) = 1 &\quad \text{for } x \in \Omega_f, \\ 0 \leqslant \varphi(x) \leqslant 1 &\quad \text{for } x \in \overline{\xi}, \\ \varphi(x) = 0 &\quad \text{for } x \in \Omega_l; \end{aligned} \tag{4}$$

see Figure 1. In the phase-field setting, the fluid region $\Omega_f$ and the lens region $\Omega_l$ are hence separated by a diffuse interface. On the diffuse interface, the material properties are interpolated with respect to the phase-field function as follows:

$$\begin{aligned} c^2 &= c_l^2 + \varphi(x)(c_f^2 - c_l^2), \\ b &= b_l + \varphi(x)(b_f - b_l), \\ k &= k_l + \varphi(x)(k_f - k_l), \end{aligned} \tag{5}$$

where we assume $c_l < c_f$, $b_l < b_f$, and $k_l < k_f$.

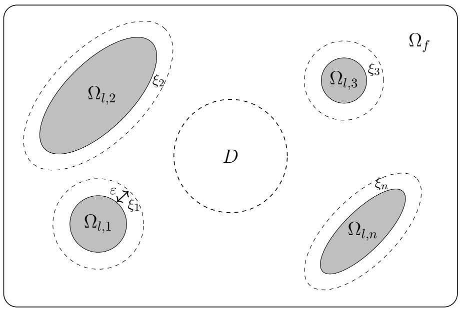


Figure 1: The acoustic lens system with a phase-field interface.

To formulate the problem, we employ a tracking-type objective and use a perimeter penalization to overcome ill-posedness of the sharp interface problem. We approximate it in the diffuse interface setting by a multiple of the Ginzburg–Landau energy $E_\varepsilon$:

$$E_\varepsilon(\varphi) = \begin{cases} \displaystyle\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\Psi(\varphi)\mathrm{d}x, & \text{if } \varphi \in H^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Here $\Psi$ is a double obstacle potential given by

$$\Psi(\varphi) = \begin{cases} \Psi_0(\varphi) & \text{if } \ 0 \leqslant \varphi \leqslant 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

with

$$\Psi_0(\varphi) = \frac{1}{2}\varphi(1-\varphi).$$

The shape optimization problem then has the following phase-field formulation:

$$\min_{(u,\varphi)} J^\varepsilon(u,\varphi) = \frac{1}{2}\int_0^T \int_D (u-u_\mathrm{d})^2 \,\mathrm{d}x\mathrm{d}s + \gamma E_\varepsilon(\varphi),$$

where $\gamma > 0$ is a weighting parameter, with

$$\begin{aligned} &\varphi \in \Phi_\mathrm{ad} = \{\varphi \in H^1(\Omega) \cap L^\infty(\Omega) : 0 \leqslant \varphi \leqslant 1 \text{ a.e.}\}, \\ &u \in U = \{u \in L^\infty(0,T;H^1(\Omega)) : u_t \in L^\infty(0,T;H^1(\Omega)), \\ &\qquad\qquad\qquad\qquad u_{tt} \in L^2(0,T;L^2(\Omega))\}, \end{aligned}$$

such that

$$\begin{cases} \alpha u_{tt} - \mathrm{div}(c^2(\varphi)\nabla u) - \mathrm{div}(b(\varphi)\nabla u_t) = 2k(\varphi)u_t^2, \\ c^2(\varphi)\dfrac{\partial u}{\partial n} + b(\varphi)\dfrac{\partial u_t}{\partial n} = g \quad \text{on} \quad \Gamma, \\ (u,u_t)|_{t=0} = (0,0), \end{cases} \tag{6}$$

is satisfied (in a weak sense) and the medium parameters satisfy (5).

The function $\varphi \in \Phi_\mathrm{ad}$ is thus the design variable with $\{x \in \Omega : \varphi(x) = 1\}$ modeling the fluid region and $\{x \in \Omega : \varphi(x) = 0\}$ the lenses.

For the well-posedness of the state problem and the corresponding adjoint problem as well the proof of the existence of a minimizer, we refer to [1].

**Theorem 1 (Optimality system)** *Let $\varphi \in \Phi_\mathrm{ad}$ be the minimizer of the optimal control problem* (2)–(6) *and $u$ and $p$ the associated state and adjoint variables, respectively. Then the functions $(u,\varphi,p) \in U \times \Phi_\mathrm{ad} \times H^1(0,T;H^1(\Omega))$ satisfy the following optimality system in the weak sense: the state problem* (6)*, the adjoint problem*

$$\begin{cases} \alpha p_{tt} - \mathrm{div}(c^2(\varphi)\nabla p) + \mathrm{div}(b(\varphi)\nabla p_t) \\ = -(4k(\varphi)u_{tt} + \alpha_{tt})p - 2(\alpha_t + 2k(\varphi)u_t)p_t) \\ \quad +(u-u_\mathrm{d})\chi_D \quad \text{in} \quad \Omega \times (0,T), \\ c^2(\varphi)\dfrac{\partial p}{\partial n} - b(\varphi)\dfrac{\partial p_t}{\partial n} = 0 \quad \text{on} \quad \Gamma, \\ (p,p_t)|_{t=T} = (0,0), \end{cases}$$

*and the gradient inequality*

$$\begin{aligned} &\gamma\varepsilon\int_\Omega \nabla\varphi\cdot\nabla(\tilde\varphi-\varphi)\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \Psi'(\varphi)(\tilde\varphi-\varphi)\,\mathrm{d}x \\ &\qquad - \int_0^T\int_\Omega (2c(\varphi)c'(\varphi)(\tilde\varphi-\varphi)\nabla u(t) \\ &\qquad\quad + b'(\varphi)(\tilde\varphi-\varphi)\nabla u_t(t))\cdot\nabla p\,\mathrm{d}x\mathrm{d}s \\ &+ \int_0^T\int_\Omega 2k'(\varphi)(\tilde\varphi-\varphi)u_t^2(t)p\,\mathrm{d}x\mathrm{d}s \geqslant 0, \ \forall\tilde\varphi \in \Phi_\mathrm{ad}. \end{aligned}$$

**Theorem 2** *Under the assumptions of the well-posedness of state and adjoint problems, the reduced cost functionals $\{j_\varepsilon\}_{\varepsilon>0}$, where $j_\varepsilon = j_\varepsilon(\varphi)$, $\Gamma$-converge in $L^1(\Omega)$ to $j_0$ as $\varepsilon \searrow 0$.*

## References


[1] H. Garcke, S. Mitra, and V. Nikolić, A phase-field approach to shape and topology optimization of acoustic waves in dissipative media, arXiv preprint arXiv:2109.13239, 2021.