Non-conforming and moving grids for the simulation of waves in viscous fluids

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Abstract

We present a Finite Element (FE) formulation for waves in viscous, compressible fluids coupled to solid bodies described by linear elasticity. In doing so, we allow for non-conforming grids based on the Nitsche-type mortaring.

Keywords: waves in viscous compressible fluids, finite elements, non-conforming grids

1 Formulation

We consider an elastic solid coupled to a viscous, compressible fluid along with a common interface as displayed in Fig. 1. The behavior of

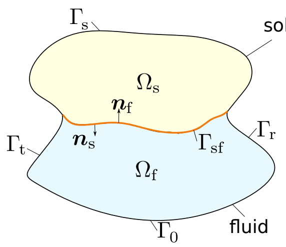

Figure 1: Simple sketch of a solid-fluid interaction problem

the solid is described by the balance of momentum and a suitable constitutive law (Hooke's law). The governing equations in the fluid domain Ω_f are the balance of mass and momentum as well as an equation of state and constitutive law (Newtonian fluid). By applying a perturbation ansatz via a splitting of the total quantities (pressure, density and velocity) into a mean part and a fluctuating one, neglecting the non-linear terms and using the linearized equation of state between density perturbation ρ and pressure p via $\rho = \frac{p}{c_0^2}$ (c donates the isentropic speed of sound), we arrive at

$$\frac{1}{c_0^2}\frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega_\mathrm{f}\,, \tag{1}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} - \nabla \cdot \boldsymbol{\sigma}_\mathrm{f} = \mathbf{0} \quad \text{in } \Omega_\mathrm{f}\,. \tag{2}$$

In (1), (2) p denotes the acoustic pressure, $\boldsymbol{v}$ the acoustic particle velocity, and $\boldsymbol{\sigma}_\mathrm{f}$ the fluid stress tensor, which computes for an isotropic Newtonian fluid by

$$\boldsymbol{\sigma}_\mathrm{f} = -p\boldsymbol{I} + \mu\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\big) \tag{3}$$

$$+(\lambda - \frac{2}{3}\mu)(\nabla \cdot \boldsymbol{v})\boldsymbol{I}\,, \tag{4}$$

where μ is the dynamic (shear) viscosity and λ the bulk viscosity. The elastic solid in Ω_s is governed by the conservation of momentum

$$\rho_\mathrm{s}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \nabla \cdot \boldsymbol{\sigma}_\mathrm{s} = \mathbf{0} \quad \text{in } \Omega_\mathrm{s}\,, \tag{5}$$

where the solid density is denoted by ρ_s, $\boldsymbol{u}$ is the mechanical displacement vector and $\boldsymbol{\sigma}_\mathrm{s}$ is the mechanical stress tensor. At the interface Γ_sf between solid and fluid, one needs to enforce the dynamic and kinematic conditions requiring continuity of traction and velocity, respectively. Traction continuity is enforced by requiring

$$\boldsymbol{\sigma}_\mathrm{s} \cdot \boldsymbol{n}_\mathrm{s} = -\boldsymbol{\sigma}_\mathrm{f} \cdot \boldsymbol{n}_\mathrm{f} \quad \text{on } \Gamma_\mathrm{sf}\,, \tag{6}$$

where $\boldsymbol{n}_\mathrm{f}$ and $\boldsymbol{n}_\mathrm{s}$ are the outer normal of the fluid and solid domain, respectively (see Fig. 1). The second interface condition is velocity continuity at the interface, which requires

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{v} \quad \text{on } \Gamma_\mathrm{sf}\,. \tag{7}$$

To obtain the Nitsche coupled formulation, we combine the weak forms of the partial differential equations (PDEs) for the viscous fluid and elastic solid and incorporate (6), (7) to arrive at

$$\int_{\Omega_\mathrm{f}} \rho_\mathrm{f}\boldsymbol{v}' \cdot \frac{\partial \boldsymbol{v}}{\partial t}\,\mathrm{d}\Omega + \int_{\Omega_\mathrm{f}} \nabla \boldsymbol{v}' : \boldsymbol{\sigma}_\mathrm{f}\,\mathrm{d}\Omega$$
$$+ \int_{\Omega_\mathrm{s}} \rho_\mathrm{s}\boldsymbol{u}' \cdot \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\,\mathrm{d}\Omega + \int_{\Omega_\mathrm{s}} \nabla \boldsymbol{u}' : \boldsymbol{\sigma}_\mathrm{s}\,\mathrm{d}\Omega$$
$$\underbrace{- \int_{\Gamma_\mathrm{sf}} (\boldsymbol{u}' - \boldsymbol{v}') \cdot \boldsymbol{\sigma}_\mathrm{s} \cdot \boldsymbol{n}\,\mathrm{d}\Gamma}_{\text{traction consistency}}$$
$$\underbrace{+ \beta \frac{p_\mathrm{e}^2}{h_\mathrm{e}} \int_{\Gamma_\mathrm{sf}} (\boldsymbol{u}' - \boldsymbol{v}') \cdot (\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{v})\,\mathrm{d}\Gamma}_{\text{penalty}} = \mathbf{0}\,. \tag{8}$$

To meet the inf-sup (Ladyzhenskaya-Babuška-Brezzi) condition, we use a one-order higher polynomial basis function for particle velocity $\boldsymbol{v}$ than that for the acoustic pressure p [1,2].

2 Validation

To validate our formulation for a viscous fluid, we consider a Stokes boundary layer generated by an infinitely long plate, which oscillates with a velocity $\hat{v}\cos(\omega t)$ in x-direction and fulfilling the solution

$$v_y = \hat{v}\,\mathrm{Re}\left\{e^{i\omega t}\, e^{-\frac{1+i}{\sqrt{2}}\sqrt{\frac{\rho\omega}{\mu}}y}\right\}. \qquad (9)$$

The computational domain is displayed in Fig. 2 for a graded mesh and the convergence behavior of the error both for $h-$ and $p-$ refinement is shown in Fig. 3. The results clearly demonstrates the superiority of p-FEM on a graded mesh.

Figure 2: Computational domain with graded mesh

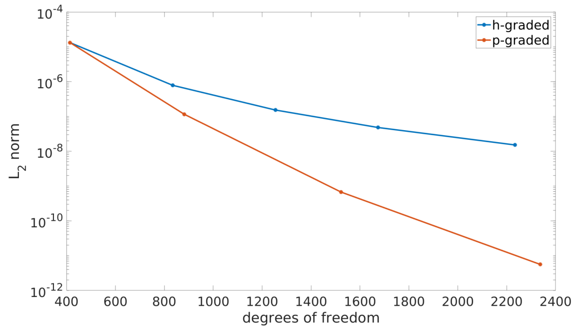

Figure 3: Convergence of h- and p-FEM on graded mesh

3 Application

As a practical application, we consider a micro-electro-mechanical system (MEMS) speaker as displayed in Fig. 4. To further decrease the computational time, we apply the linearized balance of mass and momentum of the viscous fluid (viscous PDEs) just in the channel and in a small region surrounding the ambient air, and then couple it to the standard wave equation again via a non-conforming grid to compute the radiated sound (see Fig. 5). The computational results are displayed in Fig. 6 both scaled for the channel part, where the viscous effect is strongly present and for the ambient air.

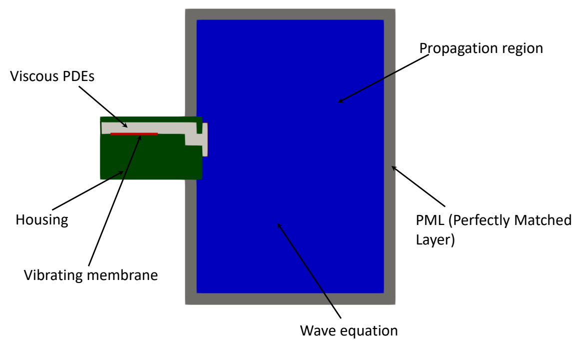

Figure 4: Computational setup for MEMS speaker

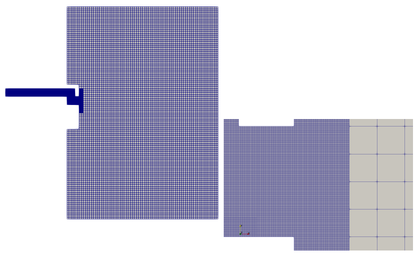

Figure 5: Computational grid

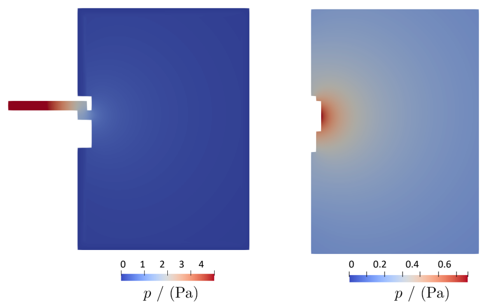

Figure 6: Computed acoustic pressure in the channel and radiated to the ambient air

Currently, we extend our formulation to include also moving structures via an ALE (Arbitrary Lagrangian - Eulerian) formulation and apply it to a MEMS speaker based on digital sound reconstruction using moving shutter gates.

References

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