Bi-Parametric Operator Preconditioning and Applications

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Abstract

We extend the operator preconditioning framework [1] to Petrov-Galerkin methods while accounting for parameter-dependent perturbations of both variational forms and their preconditioners, as occurs when performing numerical approximations. By considering different perturbation parameters for the original form and its preconditioner, our bi-parametric abstract setting leads to robust and controlled schemes. For Hilbert spaces, we derive exhaustive linear and super-linear convergence estimates for iterative solvers, such as h-independent convergence bounds, when preconditioning with low-accuracy or, equivalently, with highly compressed approximations.

Keywords: Operator preconditioning, Galerkin methods, Numerical approximation, Iterative linear solvers

1 Introduction

In this note, we extend the framework of operator preconditioning (OP) from Bubnov-Galerkin to general Petrov-Galerkin methods (OP-PG) as well as analyze the effects of numerical perturbations in iterative solvers. In this regard, we provide estimates for spectral and Euclidean condition numbers with details found in [2]. Next, we consider parameter-dependent perturbed problems and introduce the *bi-parapmetric* OP paradigm used, for example, for fast Calderón preconditioning [3, 6]. This allows for explicit bounds on spectral and Euclidean condition numbers with respect to perturbations. We further deduce linear convergence results for GMRES(m) when working on Hilbert spaces.

2 Bi-parametric Operator Preconditioning

Let X, Y, V and W be reflexive Banach spaces and let $\mathsf{a} \in \mathcal{L}(X\times Y;\mathbb{C})$ be a continuous complex sesqui-linear form, with induced operator A and norm $\|\mathsf{a}\|$. Similarly for $\mathsf{c} \in \mathcal{L}(V \times W;\mathbb{C})$, $\mathsf{n} \in \mathcal{L}(V \times Y;\mathbb{C})$, $\mathsf{m} \in \mathcal{L}(X \times W;\mathbb{C})$. For a linear form $b \in Y'$, the *weak continuous problem* is: seek $u \in X$ such that

$$\mathsf{a}(u, v) = b(v), \quad \forall\, v \in Y. \tag{1}$$

Given an index $h > 0$, we introduce finite-dimensional *conforming spaces*, i.e. $X_h \subset X$ and $Y_h \subset Y$, and assume that $\dim(X_h) = \dim(Y_h) =: N$, with $N \to \infty$ as $h \to 0$. Same occurs for $V_h \subset V$ and $W_h \subset W$. The counterpart of (1) is the *weak discrete problem*: find $u_h \in X_h$ such that

$$\mathsf{a}(u_h, v_h) = b(v_h), \quad \forall\, v_h \in Y_h. \tag{2}$$

We build the stiffness Galerkin matrix and right-hand side

$$\mathbf{A} := (\mathsf{a}(\varphi_j, \phi_i))_{i,j=1}^N, \quad \mathbf{b} := (b_h(\phi_i))_{i=1}^N,$$

$\mathrm{span}\{\varphi_i\}_{i=1}^N = X_h$, $\mathrm{span}\{\phi_i\}_{i=1}^N = Y_h$. For $\mathrm{span}\{\psi_i\}_{i=1}^N = V_h$, $\mathrm{span}\{\xi_i\}_{i=1}^N = W_h$, we have

$$\mathbf{C} := (\mathsf{a}(\psi_j, \xi_i))_{i,j=1}^N,\ \mathbf{N} := (\mathsf{n}(\varphi_j, \xi_i))_{i=1}^N$$

and $\mathbf{M} := (\mathsf{n}(\psi_j, \phi_i))_{i=1}^N$.

We are ready to give a notion of admissible perturbations needed for the ensuing analysis.

Definition 1 ((h,ν)-perturbation) *Let $\nu \in [0, 1)$ and $h > 0$ be given. We say that $\mathsf{a}_\nu \in \mathcal{L}(X \times Y;\mathbb{C})$ is a (h,ν)-perturbation of a if it belongs to the set $\Phi_{h,\nu}(\mathsf{a})$:*

$$\begin{aligned}&\mathsf{a}_\nu \in \Phi_{h,\nu}(\mathsf{a}) \iff \\ &\quad \gamma_{\mathsf{A}}^{-1} |\mathsf{a}(u_h, v_h) - \mathsf{a}_\nu(u_h, v_h)| \le \nu \|u_h\|_X \|v_h\|_Y, \\ &\quad \forall\, u_h \in X_h,\ \forall\, v_h \in Y_h.\end{aligned}$$

Likewise, we define $b_\nu \in Y'$ as being a (h,ν)-perturbation of b.

Inspired by [1], we state the preconditioned version of the operator equation for PG Galerkin: seek $u \in X$ such that

$$(\!(\mathrm{CA})\!): \quad \mathsf{PA}u = \mathsf{P}b, \quad \text{with} \quad \mathsf{P} := \mathsf{M}^{-1}\mathsf{C}\mathsf{N}^{-1}.$$

Its perturbed matrix version reads: find $\mathbf{u}_\nu \in \mathbb{C}^N$ such that

$$(\!(\mathrm{CA})\!)_{\mu,\nu}: \quad \mathbf{P}_\mu \mathbf{A}_\nu \mathbf{u}_\nu = \mathbf{P}_\mu \mathbf{b}_\nu,$$

with $\mathbf{P}_\mu := \mathbf{M}^{-1}\mathbf{C}_\mu\mathbf{N}^{-1}$.

Theorem 2 (Bi-Parametric OP-PG [2]) *For the perturbed problem* $(\!(\mathrm{CA})\!)_{\mu,\nu}$ *for* $\mu, \nu \in [0, 1)$ *and* $h > 0$*, the spectral condition number is bounded as*

$$\kappa_S(\mathbf{P}_\mu \mathbf{A}_\nu) \leq \mathrm{K}_\star \left(\frac{1+\mu}{1-\mu}\right)\left(\frac{1+\nu}{1-\nu}\right) =: \mathrm{K}_{\star,\mu,\nu}\,.$$

with $\mathrm{K}_\star$ *the original estimate for OP in [1].*

3 Iterative Solvers Performance: Hilbert space setting

Consider $X \equiv H$ with H being a Hilbert space with inner product $(\cdot,\cdot)_H$ and $\|\cdot\|_H = \sqrt{(\cdot,\cdot)_H}$. We introduce the H-field of values $\mathcal{V}_H(\cdot)$ [4].

Assumption 1 *For* $(\!(\mathrm{CA})\!)_{\mu,\nu}$ *with* $X := H$ *being a Hilbert space, assume that there holds that*

$$\frac{\gamma_{\mathsf{C}_\mu}\gamma_{\mathsf{A}_\nu}}{\|\mathsf{m}\|\|\mathsf{n}\|} \leq \mathcal{V}_H(\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu}) \quad \textit{and}$$
$$\frac{\gamma_{\mathsf{M}}\gamma_{\mathsf{N}}}{\|\mathsf{c}_\mu\|\|\mathsf{a}_\nu\|} \leq \mathcal{V}_H((\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu})^{-1}).$$

With this, we can apply the linear convergence results for GMRES to $(\!(\mathrm{CA})\!)_{\mu,\nu}$ [2].

Theorem 3 *Consider* $(\!(\mathrm{CA})\!)_{\mu,\nu}$ *along with Assumption 1. Then, the numerical radius for the weighted GMRES(m) for* $1 \leq k, m \leq N$ *is bounded as*

$$\Theta_k^{(m)} \leq \left(1 - \frac{1}{\mathrm{K}_{\star,\mu,\nu}}\right)^{\frac{1}{2}}. \tag{3}$$

4 Fast Calderón Preconditioning

We test our ideas by using the bi-parametric framework to the Calderón preconditioned EFIE [5] for a complex shape [6]. Rough approximations are implied by the $\mathcal{H}$-matrix tolerance and quadrature rules. Besides providing similar convergence results for the Euclidean GMRES(m) (see Fig. 1), one also observes significant gains in computational time and memory usage.

5 Conclusion

For general Petrov-Galerkin methods, we considered their operator preconditioning and introduced the novel bi-parametric framework. Several results were derived including bounds in Euclidean norm for the convergence of iterative solvers when preconditioning, with GMRES as a reference. These results pave the way toward new paradigms for preconditioning, as they allow to craft robust preconditioners, better understand the efficiency of existing ones and relate them to experimental results.

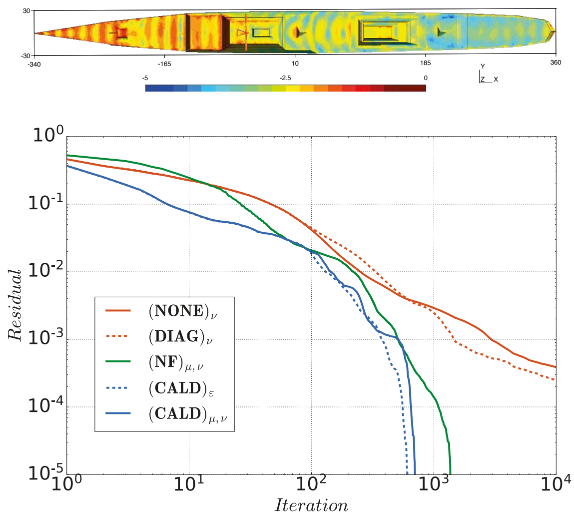

Figure 1: Destroyer for $N = 108{,}570$ dofs: Squared current density (top) and GMRES(1,500) iterations (bottom). Remark that the rough approximation for Calderón preconditioner (blue dashed line) behaves similarly to its uncompressed version (blue line).

References

[1] R. Hiptmair, Operator Preconditioning, *Computers and Mathematics with Applications* **5** (2006), pp. 699–706.

[2] P. Escapil-Inchauspé, C. Jerez-Hanckes, Bi-parametric Operator Preconditioning, *Computers and Mathematics with Applications* **102** (2021), pp. 220–232

[3] I. Fierro, C. Jerez-Hanckes, Fast Calderón Preconditioning for Helmholtz Equations, *Journal of Computational Physics* **409** (2020), 109355.

[4] M. Benzi, *Exploiting Hidden Structure in Matrix Computations: Algorithms and Applications*, Springer, 2016.

[5] F. Andriulli, K. Cools, H. Bagci, F. Olyslager, A. Buffa, S. Christiansen and E. Michielssen, A Multiplicative Calderón Preconditioner for the Electric Field Integral Equation, *IEEE Transactions on Antennas and Propagation* **8** (2008), pp. 2398–2412.

[6] P. Escapil-Inchauspé, C. Jerez-Hanckes, Fast Calderón Preconditioning for the Electric Field Integral Equation, *IEEE Transactions on Antennas and Propagation*, **67** (2019), pp. 2555–2564.