# Scattering by a Body in a Pipe 

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#### Abstract

The scattering of long waves by an obstacle in a rigid pipe is considered. Various approximations are obtained. These involve the blockage coefficient, which is a far-field quantity associated with potential flow along the pipe.


Keywords: acoustics, matched asymptotics

## 1 Introduction

Many years ago, the first author co-authored a paper [4] in which matched asymptotics were used for a two-dimensional waveguide problem. The waveguide occupies the strip $|x|<a,-\infty<$ $z<\infty$ in the $x z$-plane. It contains a bounded obstacle $B$ (with boundary $S$ ) which is assumed to be symmetric about the line $z=0$. A lowfrequency plane wave is incident on $B$, and is partly reflected and partly transmitted. Thus, with $\mathrm{e}^{-\mathrm{i} \omega t}$ suppressed, the solution $u$ satisfies

$$
u \sim\left\{\begin{array}{cl}
\mathrm{e}^{\mathrm{i} k z}+R \mathrm{e}^{-\mathrm{i} k z}, & z \rightarrow-\infty,  \tag{1}\\
T \mathrm{e}^{\mathrm{i} k z}, & z \rightarrow \infty,
\end{array}\right.
$$

together with $\left(\nabla^{2}+k^{2}\right) u=0$ in the waveguide, $\partial u / \partial x=0$ at $x= \pm a$ (the boundary is soundhard (rigid)) and a boundary condition on $S$. The problem is to estimate the reflection coefficient $R$ and the transmission coefficient $T$, assuming that $k a \ll 1$.

Intuitively, we might expect Laplace's equation $\nabla^{2} u=0$ to be appropriate near $B$. Indeed, this is part of the story, but not the whole story. Historically, Lamb followed the intuitive path for a hard circular $S$, and he obtained a wrong estimate for $R$, whereas Twersky obtained the correct result some years later; see [4] for details and references.

In [4], two inner problems are identified, one for $\nabla^{2} u=0$, and one for Poisson's equation, $\nabla^{2} u=1$. The first of these requires the determination of a certain constant, known as the blockage coefficient. This constant is needed in the matching between the outer expansion of the inner solution, and the inner expansion of the outer (wave-like) solution.

More recently, the authors have generalized the basic methods from [4] (which apply to twodimensional problems with sound-hard obstacles) to three-dimensional problems in which long waves along a rigid cylindrical waveguide interact with an obstacle $B$; the boundary of $B, S$, can be hard (rigid, Neumann condition) or soft (Dirichlet condition) [5]. Here, we describe our recent work on rigid obstacles, with some remarks on soft obstacles in section 4.

## 2 A rigid body in a rigid pipe

Consider a cylindrical pipe of infinite length containing an obstacle $B$. Both the pipe's wall and the boundary of $B$ are rigid. Assume that $B$ is symmetric about the plane $z=0$, where the $z$-axis is along the pipe. The solution $u$ satisfies (1). In [5], estimates for $R$ and $T$ are obtained:

$$
R=D_{\mathrm{s}}-D_{\mathrm{a}}, \quad T=1+D_{\mathrm{s}}+D_{\mathrm{a}}
$$

where

$$
D_{\mathrm{s}}=\frac{\mathrm{i} \kappa M}{1-\mathrm{i} \kappa M}, \quad D_{\mathrm{a}}=\frac{\mathrm{i} \kappa L}{1-\mathrm{i} \kappa L},
$$

$\kappa=k a, a$ is defined by equating the pipe's crosssectional area to $\pi a^{2}, M=-|B| /\left(2 \pi a^{3}\right)$ and $|B|$ is the volume of the body $B$. The quantity $L$ is the blockage coefficient; see section 3 .

The approximations found for $R$ and $T$ satisfy known constraints, $|R|^{2}+|T|^{2}=1$ and $R T^{*}+R^{*} T=0$, exactly, where the $*$ denotes complex conjugation. They also agree with rigorous long-wave asymptotic approximations for a sphere in a pipe of circular cross-section [2].

## 3 The blockage coefficient

The blockage coefficient $L$ is a dimensionless constant defined uniquely by solving a boundaryvalue problem for a potential $\Phi: \nabla^{2} \Phi=0$ in the pipe, the normal derivative $\partial \Phi / \partial n=0$ on the pipe's wall and on $S$, and $\Phi$ satisfies

$$
\begin{equation*}
\Phi=(z / a) \pm L+o(1) \quad \text { as } z \rightarrow \pm \infty ; \tag{2}
\end{equation*}
$$

the far-field conditions (2) eliminate an arbitrary additive constant. We see that $\Phi$ is the velocity potential for uniform flow along the pipe; $L$ can be regarded as giving a measure of the obstruction, or blockage, to the flow caused by the presence of the body in the pipe.

In order to use our simple approximations for $R$ and $T$, we need $L$, but we do not want to solve the full boundary-value problem for $\Phi$ : it is complicated! If the pipe is a circular cylinder (a tube) we could contemplate solving a boundary integral equation over $S$ using Green's function for the tube [3].

We do have Hurley's useful exact formula [1]

$$
\begin{equation*}
L=\frac{1}{2 \pi a^{2}} \int_{S} \Phi \frac{\partial z}{\partial n} \mathrm{~d} S, \tag{3}
\end{equation*}
$$

where the normal vector on $S$ points outwards. This formula relates the far-field quantity $L$ to $\Phi$ evaluated on $S$, a near-field quantity.

If $B$ is small (with diameter small compared to the diameter of the pipe's cross-section), we could replace $\Phi$ in (3) by the corresponding potential for flow past $B$ in an infinite fluid. Doing this works well; see [5] for examples.

For thin objects in a tube, such as a rigid disc in the plane $z=0$, approximations can be developed using dual integral equations when the centre of the disc is on the axis of the tube. Of interest are situations where the disc almost blocks the tube, leaving a small gap between the edge of the disc and the tube wall; see [6].

Suppose now that $B$ is slender and aligned with the flow; for example, $B$ could be a prolate spheroid with its axis along the $z$-axis. Such geometries recall Webster's horn equation: for waves along a rigid pipe of slowly-varying crosssection (the horn), the three-dimensional wave equation may be approximated by

$$
\begin{equation*}
\frac{1}{A(z)} \frac{\partial}{\partial z}\left(A(z) \frac{\partial u}{\partial z}\right)=\frac{1}{c^{2}} \frac{\partial^{2} u}{\partial t^{2}}, \tag{4}
\end{equation*}
$$

where $A(z)$ is the cross-sectional area at station $z$; the derivation of (4) assumes that the horn is rigid, with $\partial u / \partial n=0$ on the boundary. We can use (4) for wave propagation along the pipe containing a slender body $B$; in this context, $A(z)$ is the area of the annular fluid cross-section. When applied to the estimation of $L$ (no dependence on $t$ ), we obtain an ordinary differential equation; solving it gives

$$
\begin{equation*}
L=\frac{1}{2 a} \int \frac{\pi a^{2}-A(z)}{A(z)} \mathrm{d} z \tag{5}
\end{equation*}
$$

note that $A(z)=\pi a^{2}$ outside $B$. It turns out that this simple formula works remarkably well, and it also gives good approximations when $B$ almost fills the waveguide; for details and comparisons, see [6].

## 4 Discussion

We have considered the reflection and transmission of long waves by an obstacle $B$ in a rigid cylindrical waveguide. For a rigid obstacle, this led us to study the blockage coefficient $L$. This far-field quantity appears in other related applications of matched asymptotic approximations. Further applications are noted in [6].

The situation for sound-soft obstacles is more complicated: one is required to calculate two additional coefficients, denoted by $P$ and $Q$ in [5]; $P$ is related to the electrostatic capacity of $B$ in the pipe.

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